Department of Mathematics
Third Annual
High School Problem Solving Contest
November 1, 2018

Solutions

1. **10 points**

   Two students attempted to solve a quadratic equation, $x^2 + bx + c = 0$. Although both students did the work correctly, the first miscopied the middle term and obtained the solution set $\{-3, 4\}$. The second student miscopied the constant term and obtained the solution set $\{-1, 5\}$. What are the correct solutions?

   Solution 1: Since $(x + 3)(x - 4) = x^2 - x - 12$ and $(x + 1)(x - 5) = x^2 - 4x - 5$, the first student was solving the equation $x^2 - x - 12 = 0$ and the second student was solving the equation $x^2 - 4x - 5 = 0$. The correct equation is $x^2 - 4x - 12 = 0$, so the correct solutions are $x = -2$ and $x = 6$.

   Solution 2: Recall Vieta's formulas: if $x = r_1$ and $x = r_2$ are two roots of $x^2+bx+c = 0$, then $b = -(r_1 + r_2)$ and $c = r_1r_2$. Since the first student got $r_1 = -3$ and $r_2 = 4$, we have $c = -12$. The second student got $r_1 = -1$ and $r_2 = 5$, so $b = -(-1 + 5) = -4$. Therefore the correct equation is $x^2 - 4x - 12 = 0$, so the correct solutions are $x = -2$ and $x = 6$.

2. **10 points**

   An unbiased coin is tossed. If the result is a head, then a pair of regular unbiased dice is rolled and the number obtained by adding the numbers on the top faces is noted down. If the result is a tail, then a card from a well-shuffled pack of eleven cards numbered $2, 3, 4, \ldots, 11, 12$ is picked and the number on the card is noted down. What is the probability that the noted number is 7 or 8?

   Solution: Let E_1 = coin toss is a head, E_2 = coin toss is a tail, A = number noted is a 7 or 8. We know that $P(E_1) = P(E_2) = \frac{1}{2}$. Also, $P(A|E_1) = P(7)+P(8) = \frac{6}{36}+\frac{5}{36} = \frac{11}{36}$, and $P(A|E_2) = \frac{2}{11}$.
   So, $P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) = \frac{11}{36} \cdot \frac{1}{2} + \frac{2}{11} \cdot \frac{1}{2} = \frac{193}{792}$.

3. **10 points**

Prove that for any integer number m, the value of

$$\frac{m}{3}+\frac{m^2}{2}+\frac{m^3}{6}$$

is also an integer.

Solution: Observe that

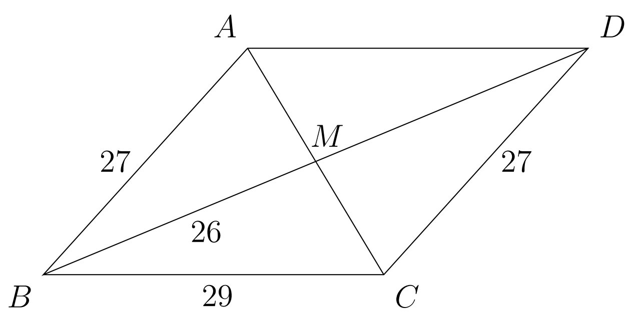
$$\begin{aligned}\frac{m}{3}+\frac{m^2}{2}+\frac{m^3}{6} &= \frac{2m+3m^2+m^3}{6}\\ &= \frac{m(2+3m+m^2)}{6}\\ &= \frac{m(m+1)(m+2)}{6}.\end{aligned}$$

The numerator is the product of three consecutive integers, therefore it is divisible by both 2 and 3, so it is divisible by 6. Thus the value of the quotient is an integer.

4. **10 points**

In $\triangle ABC$, $AB = 27$, $BC = 29$, and median $BM = 26$. Find the area of $\triangle ABC$.

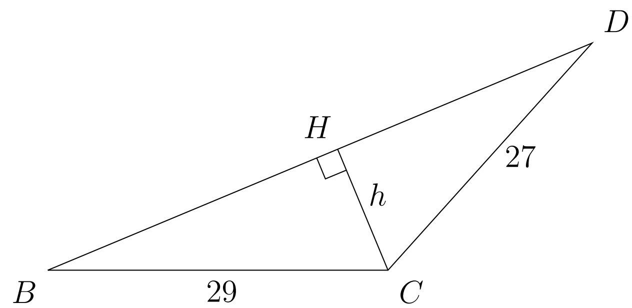
Solution 1: Draw parallelogram $ABCD$. Then $CD = AB = 27$ and $BD = 2BM = 52$.



The area of $\triangle ABC$ is half of the area of $ABCD$, and so is the area of $\triangle BCD$. Using Heron's formula (with $s = \frac{BC+CD+BD}{2}$), we have that the area of $\triangle BCD$ is

$$\begin{aligned}\sqrt{s(s-BC)(s-CD)(s-BD)} &= \sqrt{54(54-29)(54-27)(54-52)}\\ &= \sqrt{54\cdot 25\cdot 27\cdot 2}\\ &= 270.\end{aligned}$$

Solution 2: Draw parallelogram $ABCD$ with $CD = AB = 27$ and $BD = 2BM = 52$ as in Solution 1. Let $CH = h$ be an altitude in $\triangle BCD$.



Then

$$\sqrt{29^2 - h^2} + \sqrt{27^2 - h^2} = BH + HD = BD = 52.$$

Solving this equation gives $h = \frac{135}{13}$, so Area$(\triangle ABC)$ = Area$(\triangle BCD) = \frac{1}{2} \cdot 52 \cdot \frac{135}{13} = 270$.

5. $\boxed{\textbf{10 points}}$

There are p points in space, no four of which are in the same plane with the exception of q (where $q < p$) points which are all in the same plane. Find the number of (distinct) planes in space each containing three of the points.

Solution: Since it takes three points to form a plane in general, we can form $\binom{p}{3}$ planes. However, since q of these points are in the same plane, we have counted this plane $\binom{q}{3}$ times. Thus, the number of distinct planes is $\binom{p}{3} - \binom{q}{3} + 1$.

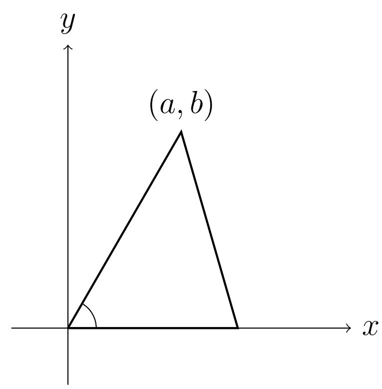
6. $\boxed{\textbf{10 points}}$

Does there exist a triangle in the xy-plane with $60°$ angle such that its vertices have integer coordinates?

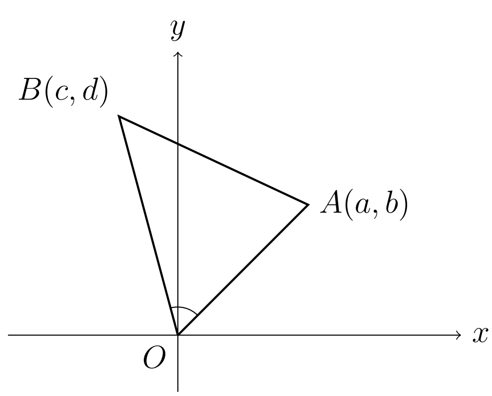
Solution:

Assume that such a triangle exists. Consider the following two cases.

Case 1: one of the sides adjacent to the $60°$ angle is parallel to one of the axes. Then without loss of generality, we can assume that the vertex of the $60°$ angle is at the origin, another vertex is on the positive x-axis, and the third vertex is (a, b), where $a, b > 0$. Then $\frac{b}{a} = \tan(60°) = \sqrt{3}$, which is impossible since $\sqrt{3}$ is irrational.



Case 2: none of the sides adjacent to the $60°$ angle is parallel to either of the axes. Then without loss of generality, we can assume that the vertex of the $60°$ angle is at the origin O, and the other vertices are at points $A = (a, b)$ and $B = (c, d)$ such that $a, b, d > 0$, and the ray OA is between the ray OB and the positive x-axis. Let α be the angle formed by OA and the positive x-axis, and let β be the angle formed by OB and the positive x-axis. Then we have $60° = \beta - \alpha$. Therefore $\sqrt{3} = \tan(60°) = \tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \dfrac{\frac{d}{c} - \frac{b}{a}}{1 + \frac{d}{c}\frac{b}{a}} = \dfrac{ad - bc}{ac + bd}$, which is again impossible since $\sqrt{3}$ is irrational.



Therefore, such a triangle does not exist.