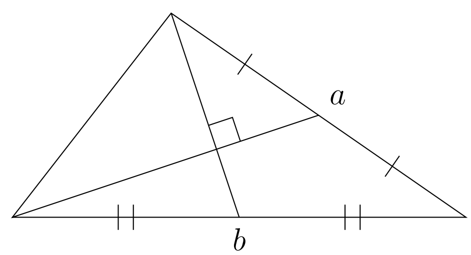
Department of Mathematics
Second Annual
High School Problem Solving Contest
November 8, 2017

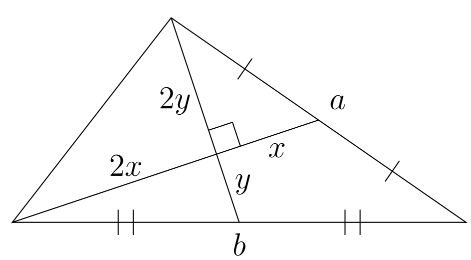
Solutions

1. **10 points**

Two sides of a triangle are a and b. The medians drawn to these sides intersect at a right angle. Find the third side of the triangle.



Solution: The intersection point of the medians divides each median into two pieces whose lengths are in ratio $2:1$. Denote these pieces $2x$, x, $2y$, and y as shown in the picture below.



Then $x^2+(2y)^2=\left(\frac{a}{2}\right)^2$ and $(2x)^2+y^2=\left(\frac{b}{2}\right)^2$. Adding these two equations gives $5x^2+5y^2=\frac{a^2+b^2}{4}$. The third side of the triangle is $\sqrt{(2x)^2+(2y)^2}=\sqrt{\frac{4}{5}(5x^2+5y^2)}=\sqrt{\frac{a^2+b^2}{5}}$.

2. **10 points**

For what values of a are both roots of the equation

$$x^2+(1-a)x+a=0$$

positive?

Solution 1: Let $f(x) = x^2 + (1-a)x + a = 0$. Both roots of the above equation are positive if and only if the x-coordinate of the vertex of the parabola $y = f(x)$ is positive, its y-coordinate is nonpositive, and $f(0) > 0$. Thus we have:

$$\frac{a-1}{2} > 0,$$

$$\left(\frac{a-1}{2}\right)^2 + (1-a)\frac{a-1}{2} + a \leq 0,$$

and

$$a > 0.$$

The first inequality is equivalent to $a > 1$, and the second to $a^2 - 6a + 1 \geq 0$. From the latter, we have $a \leq 3 - 2\sqrt{2}$ or $a \geq 3 + 2\sqrt{2}$. Combining with $a > 1$, we have $a \geq 3 + 2\sqrt{2}$.

Solution 2: The roots of the equation are $\frac{-(1-a)\pm\sqrt{(1-a)^2-4a}}{2} = \frac{-(1-a)\pm\sqrt{a^2-6a+1}}{2}$. The roots are real if and only if $a^2-6a+1 \geq 0$, i.e., if and only if $a \leq 3-2\sqrt{2}$ or $a \geq 3+2\sqrt{2}$. Both roots are positive if and only if $\frac{-(1-a)-\sqrt{a^2-6a+1}}{2} > 0$. This inequality is equivalent to $a - 1 > \sqrt{a^2 - 6a + 1}$, i.e., $a > 1$ and $a^2 - 2a + 1 > a^2 - 6a + 1$. Thus the solution is $a \geq 3 + 2\sqrt{2}$.

Solution 3: The roots, x_1, x_2, are positive, real numbers if and only if $x_1 + x_2 > 0$, $x_1 \cdot x_2 > 0$ and x_1, x_2 are real.

Note that $x_1 + x_2 > 0 \iff a > 1$, $x_1 \cdot x_2 > 0 \iff a > 0$, and x_1, x_2 are real $\iff B^2 - 4AC = a^2 - 6a + 1 \geq 0$. It follows that both roots are positive if and only if $a \geq 3 + 2\sqrt{2}$.

3. **10 points**

Each face of a cube is painted red or blue, each with a probability of $1/2$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all of the same color?

Solution: First, there are 2^6 possible colorings. The property is satisfied if 5 or 6 faces are of the same color. This can be done in $6+1 = 7$ ways for a given color. If 4 faces are of the same color, then the property is satisfied for 3 such cubes for a given color of the 4 faces (3 pairs of opposite faces can be colored different than these 4 faces). Finally, if 3 faces are of the same color, then the property cannot be satisfied. Thus, the favorable outcomes are $2(7 + 3) = 20$ in number. This gives a probability of $20/64 = 5/16$.

4. **10 points**

Let n be a natural number. An ant wants to crawl a path of length $2n$ along the grid lines that starts and ends at the origin. Prove that there are $\binom{2n}{n}^2$ ways to do this, where $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$ is a binomial coefficient.

Solution: Consider the $2n$ unit segments of the path, each of which goes either up, down, to the right, or to the left. Let U, D, R, and L denote the number of segments that go up, down, to the right, and to the left, respectively. Since the ant wants to end at the same place where it starts, we must have $U = D$ and $R = L$. Therefore $U + R = n$ and $U + L = n$. There are $\binom{2n}{n}$ ways to choose the segments that belong to the set of those that go up or to the right. There are also $\binom{2n}{n}$ ways to choose the segments that belong to the set of those that go up or to the left. Observe that a path is uniquely determined by these two sets. Therefore there are $\binom{2n}{n}^2$ such paths.

5. **10 points**

The graph of

$$2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$$

is an ellipse in the first quadrant of the xy-plane. Let a and b be the minimum and maximum values of y/x, respectively, over all points (x, y) on the ellipse. What is the value of $a + b$?

Solution: The quotient y/x represents the slope of a line through the origin and passing through the point (x, y). So, the minimum and maximum values of this fraction over all points on the ellipse happen when the line $y = mx$ becomes tangent to the ellipse. Setting $y = mx$ in the equation of the ellipse yields:

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0.$$

For the line to be tangent, this equation must have a unique root, and hence, the discriminant must be zero. That is,

$$-80m^2 + 280m - 199 = 0.$$

This implies

$$a + b = m_1 + m_2 = 280/80 = 7/2.$$

6. **10 points**

Given that 2^{2017} is a 608-digit number with first digit 1, how many elements of the set $S = \{2^0, 2^1, 2^2, \ldots, 2^{2016}\}$ have a first digit of 4?

Solution: For each n, there must be exactly one power of 2 that has exactly n digits with first digit 1. Thus, S contains exactly 607 elements that have a first digit of 1. For each number of the form 2^k with first digit of 1, 2^{k+1} must have a first digit of either 2 or 3, and 2^{k+2} must have a first digit of either 4, 5, 6, or 7. Thus, there are exactly 607 elements in S with first digit in $\{2, 3\}$ and 607 more elements with first digit in $\{4, 5, 6, 7\}$. So, by complement, there must be $2017 - 3 \times 607 = 196$ elements in S with first digit 8 or 9. But the only way for 2^k to have a first digit of 8 or 9 is for 2^{k-1} to have a first digit of 4. Thus, there are 196 elements in S that have first digit of 4.