Department of Mathematics
Fourth Annual
Problem Solving Contest
November 14, 2018

Solutions

1. **10 points**

   Numbers a, b, and c are chosen randomly and independently from the set

   $$\{n \in \mathbb{Z} \mid -5 \le n \le 5\}.$$

   What is the probability that the function

   $$f(x) = \begin{cases} ax + b & \text{if } x \le c \\ x^2 & \text{if } x > c \end{cases}$$

   is differentiable everywhere?

   Solution:

   Function $f(x)$ defined above is differentiable everywhere except possibly at $x = c$. It is differentiable at $x = c$ if both the values and the derivatives of $ax + b$ and x^2 agree at $x = c$. That is, if $ac + b = c^2$ and $a = 2c$. Substituting $2c$ for a in the first equation and solving for b gives $b = -c^2$. So, once the value of c is chosen, the values of a and b are determined uniquely. From the set $\{n \in \mathbb{Z} \mid -5 \le n \le 5\}$, only the values $c = 0$, $c = \pm 1$, and $c = \pm 2$ produce a and b that are also in this set. Therefore the probability of such a choice of c, a, and b is $\frac{5}{11} \cdot \frac{1}{11} \cdot \frac{1}{11} = \frac{5}{1331}$.

2. **10 points** Prove that for every positive integer n

   $$n! \le \left(\frac{n+1}{2}\right)^n.$$

   Solution: For any positive integer n, apply the AM-GM inequality to the set $\{1, 2, \ldots, n\}$ of positive integers. This gives

   $$\sqrt[n]{1 \cdot 2 \cdot \cdots \cdot n} \le \frac{1 + 2 + \cdots + n}{n}$$

   which becomes

   $$n! \le \left(\frac{n+1}{2}\right)^n.$$

3. **10 points** Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be such that

$$\sum_{k=1}^{n} \frac{a_k}{4k+1} = 0.$$

Prove that the function

$$f(x) = \sum_{k=1}^{n} a_k \cos((4k+1)x), \ \ x \in \mathbb{R}$$

has at least one zero in the interval $(0, \pi/2)$.

Solution: Consider the antiderivative of f

$$F(x) = \sum_{k=1}^{n} \frac{a_k}{4k+1} \sin((4k+1)x), \ \ x \in \mathbb{R}.$$

The function F satisfies the conditions of Rolle's Theorem on $[0, \pi/2]$:

(1) continuous on $[0, \pi/2]$,
(2) differentiable on $(0, \pi/2)$, and
(3) $F(0) = 0 = \sum_{k=1}^{n} \frac{a_k}{4k+1} = F(\pi/2)$,

and hence, its derivative f has at at least one zero in the interval $(0, \pi/2)$.

4. **10 points**

Consider the collection of all three element subsets of the set $\{1, 2, 3, \ldots, 299, 300\}$. Determine the number of these subsets for which the sum of the three elements is a multiple of 3.

Solution:

For $0 \le j \le 2$, let $A_j = \{x \mid 1 \le x \le 300, x \equiv j \pmod 3\}$. Then, the desired subsets of the form $\{a, b, c\}$ in the problem arise only from two cases:

(a) All of a, b, c are from A_0, or A_1, or A_2; or
(b) One of a, b, c is from A_0, another from A_1, and the third from A_2.

Thus, the desired number of such subsets is

$$3\binom{100}{3} + 100^3 = 1{,}485{,}100.$$

5. **10 points** Let F_n denote the n^{th} Fibonacci number (with $F_0 = F_1 = 1$). Show that the product of any four consecutive Fibonacci numbers ($F_n F_{n+1} F_{n+2} F_{n+3}$) is the area of a Pythagorean triangle (right triangle whose sides have integer lengths).

Solution: Suppose $F_{n+1} = a$ and $F_{n+2} = b$. Then $F_n = b - a$ and $F_{n+3} = b + a$. Since $(b^2 - a^2)^2 + (2ab)^2 = (b^2 + a^2)^2$, we have that $b^2 - a^2$, $2ab$, $b^2 + a^2$ form the sides a Pythagorean triangle with area $ab(b^2 - a^2) = F_n F_{n+1} F_{n+2} F_{n+3}$.

6. **10 points**

Let $\{a_1, a_2, \ldots, a_{100}\}$ be a set of 100 integer numbers. Prove that this set contains a subset in which the sum of all elements is divisible by 100.

Solution:

Consider the subsets $\{a_1\}$, $\{a_1, a_2\}$, $\ldots$, $\{a_1, a_2, \ldots, a_{100}\}$. Let S_1, S_2, $\ldots$, S_{100} be the sums of their elements, respectively. If all $S_i \bmod 100$ are distinct, then one of them is 0, so we have a required subset. If $S_i \bmod 100 = S_j \bmod 100$ for some $i < j$, then the subset $\{a_{i+1}, \ldots, a_j\}$ has the sum of elements divisible by 100.