Department of Mathematics
Fourth Annual
Problem Solving Contest
November 14, 2018

Name: ______________________________________

Email: ______________________________________

1. $\boxed{\textbf{10 points}}$

   Numbers a, b, and c are chosen randomly and independently from the set

   $$\{n \in \mathbb{Z} \mid -5 \le n \le 5\}.$$

   What is the probability that the function

   $$f(x) = \begin{cases} ax + b & \text{if } x \le c \\ x^2 & \text{if } x > c \end{cases}$$

   is differentiable everywhere?

2. $\boxed{\textbf{10 points}}$

   Prove that for every positive integer n

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

3. $\boxed{\textbf{10 points}}$

Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be such that

$$\sum_{k=1}^{n} \frac{a_k}{4k+1} = 0.$$

Prove that the function

$$f(x) = \sum_{k=1}^{n} a_k \cos((4k+1)x), \;\; x \in \mathbb{R}$$

has at least one zero in the interval $(0, \pi/2)$.

4. $\boxed{\textbf{10 points}}$

   Consider the collection of all three element subsets of the set $\{1, 2, 3, \ldots, 299, 300\}$. Determine the number of these subsets for which the sum of the three elements is a multiple of 3.

5. $\boxed{\textbf{10 points}}$

   Let F_n denote the n^{th} Fibonacci number (with $F_0 = F_1 = 1$). Show that the product of any four consecutive Fibonacci numbers ($F_n F_{n+1} F_{n+2} F_{n+3}$) is the area of a Pythagorean triangle (right triangle whose sides have integer lengths).

6. $\boxed{\textbf{10 points}}$

   Let $\{a_1, a_2, \ldots, a_{100}\}$ be a set of 100 integer numbers. Prove that this set contains a subset in which the sum of all elements is divisible by 100.