Department of Mathematics
Third Annual Problem Solving Contest
November 16, 2017

Solutions

1. **10 points**

   Real numbers a and b are chosen randomly and independently in the interval $[-1, 1]$. Find the probability that the line $y = ax + b$ and the parabola $y = x^2$ intersect.

   Solution: The line and the parabola intersect if the equation $x^2 = ax + b$ has real solutions, i.e. if the discriminant of the equation $x^2 - ax - b = 0$ is nonnegative: $a^2 + 4b \geq 0$. This inequality is equivalent to $b \geq -\frac{a^2}{4}$, thus we have to find the ratio of the area of the region given by $b \geq -\frac{a^2}{4}$, $-1 \leq a \leq 1$, $-1 \leq b \leq 1$, to the area of the whole square $-1 \leq a \leq 1$, $-1 \leq b \leq 1$. The area of the above region is $\int_{-1}^{1} \left(1 + \frac{a^2}{4}\right) da = \frac{13}{6}$, and the area of the whole square is 4. Thus, the probability of the discriminant being nonnegative is $\frac{13}{24}$.

2. **10 points**

   Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$, and $a + b + c = 2$. Prove that
   $$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

   Solution: Let $x = 1 - a$, $y = 1 - b$, and $z = 1 - c$. The desired inequality is then equivalent to
   $$\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \geq 8,$$
   which, in turn, is equivalent to
   $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$
   Using the AM-HM inequality and $x + y + z = 1$, this inequality follows.

3. **10 points**

   A unit cube is projected onto a plane. Prove that the sum of the squares of the lengths of the projections of all its edges is equal to 8.

   Solution: Choose the coordinate system so that points $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, and $D(0,0,1)$ are four of the cube's vertices. Let (x, y, z) be a unit vector normal to the plane, and let α, β, and γ be the angles that this vector forms with the edges AB, AC, and AD, respectively. Then $x = \cos\alpha$, $y = \cos\beta$, and $z = \cos\gamma$, therefore, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The lengths of the projections of the edges AB, AC, and AD onto the plane are $\sin\alpha$, $\sin\beta$, and $\sin\gamma$, respectively, so the sum of their squares is $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2$. Since the cube has a total of 4 edges parallel to each of AB, AC, and AD, the sum of the squares of the lengths of all their projections is 8.

4. **10 points**

Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

Solution: For each n, $0 \le n \le 10$, let A_n denote the set of integers between 1 and 100 which leave a remainder of n upon division by 11. Then A_1 has 10 elements, and A_n, $n \ne 1$, has 9 elements each. If $\{a, b\}$ is any two-element subset of $\{1, 2, 3, \ldots, 100\}$, then 11 divides $a + b$ if and only if either both a, b are in A_0 or else a is in A_k and b is in A_{11-k} for some k, $1 \le k \le 10$.

Consider any set B with 48 elements. If B contains two elements from the set A_0, then we are done. Similarly, if B contains an element from A_k and another from A_{11-k}, $1 \le k \le 10$ then again, their sum is divisible by 11. Thus, B can contain one element from A_0, 10 from A_1, and 9 from the sets A_k for some 4 values of $k (\ne 10)$, say k_1, k_2, k_3, k_4, no two of which add up to 11.

But these account only for 47 elements. Hence, there must be an element which is either in A_0 or A_{10} or in $A_{11-k_j}, 1 \le j \le 4$. Thus, we can always find either two elements a, b that are both in A_0 or one element a in A_k and another element b in A_{11-k} for any subset B with 48 elements. In either case, 11 divides $a + b$.

5. **10 points**

Determine all non-negative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

Solution: The given equation is equivalent to

$$(xy - 6)^2 + 13 = (x + y)^2,$$

which, in turn, is equivalent to

$$13 = \big((x + y) + (xy - 6)\big)\big((x + y) - (xy - 6)\big).$$

Since 13 is prime, the only possible (integer) factors are ± 1, ± 13. So, we have four cases:

(a) $(x + y) + (xy - 6) = 1, \quad (x + y) - (xy - 6) = 13,$

(b) $(x + y) + (xy - 6) = 13, \quad (x + y) - (xy - 6) = 1,$

(c) $(x + y) + (xy - 6) = -1, \quad (x + y) - (xy - 6) = -13,$

(d) $(x + y) + (xy - 6) = -13, \quad (x + y) - (xy - 6) = -1.$

Case (a) yields solutions $(7, 0)$ and $(0, 7)$. Case (b) yields $(3, 4)$ and $(4, 3)$. Cases (c) and (d) do not have any non-negative solutions. Thus, the only non-negative integral solutions are $(7, 0)$, $(0, 7)$, $(3, 4)$, and $(4, 3)$.

6. **10 points**

Prove that the sequence

$$x_n = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\dots \sqrt{2 + \sqrt{2}}}}}} \quad (n \text{ roots}),\ n \in \mathbb{N},$$

is convergent and find $\lim_{n\to\infty} x_n$.

Solution: The sequence is defined by the recursive relation

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2 + x_n},\ n \geq 1. \tag{1}$$

It can be shown via induction that the sequence is bounded:

$$0 \leq x_n \leq 2$$

and increasing.

Hence, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} x_n \in [0, 2]$$

exists.

Passing to the limit in (1) as $n \to \infty$, we arrive at the equation

$$x = \sqrt{2 + x},$$

solving which, in view of the non-negativity of the limit, we find $x = 2$.