2023
Leap Frog Relay Grades 9-10

Part I — Problems 1-10

DO NOT OPEN PACKET UNTIL INSTRUCTED.

Things to remember:

1. Use a #2 pencil to fill out your Scantron form as follows:

   NAME: (Both team members' full names)

   SUBJECT: (Team's full **school** name)

   TEST NO.: Leap Frog 9-10

2. No calculators, slide rules, or other such devices are allowed. You may not consult notes, books, or the internet.

3. Students must work individually during the first hour. **No communication** is allowed during the first hour. **CONTESTANTS CAUGHT CHEATING WILL BE DISQUALIFIED.**

4. Each correct answer = 4; each incorrect answer = -1; blank = 0.

5. During the second hour you may trade papers and communicate quietly with your partner.

6. You may write on this packet, but your work will not be evaluated. Be sure to mark your team's final answers on the Scantron form.

2023 Leap Frog Relay Grades 9-10 Part I — Problems 1-10

No calculators allowed

Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. The graphs of $y^6 = x^2$ and $|y| = |x|$ are plotted on the Cartesian plane. How many different regions do these graphs divide the plane into?

(a) 8

(b) 11

(c) 12

(d) 13

(e) None of these

2. What is the smallest positive whole number that can be divided by each of the whole numbers from 1 to 10 without any remainder?

(a) 1260

(b) 1680

(c) 5040

(d) 2520

(e) None of these

3. What is the last digit (ones place) in the sum $1^1 + 2^2 + 3^3 + ... + 10^{10}$?

(a) 7

(b) 1

(c) 9

(d) 4

(e) None of these

4. An integer can be represented in binary form by writing it as a sum of powers of 2. For example,

$$\begin{aligned} 13 &= 8 + 4 + 1 \\ &= (a_4 \times 2^4) + (a_3 \times 2^3) + (a_2 \times 2^2) + (a_1 \times 2^1) + (a_0 \times 2^0). \end{aligned}$$

The coefficients next to the powers of two, 1's and 0's, are used to represent a number in binary. Hence $13 = (a_4a_3a_2a_1a_0)_2 = (10101)_2$.

What is 37 in binary?

(a) $(100001)_2$

(b) $(101011)_2$

(c) $(100101)_2$

(d) $(100010)_2$

(e) None of these

5. How many different integers can be represented with a binary string of the form $(a_4a_3a_2a_1a_0)_2$?

(a) 15

(b) 64

(c) 31

(d) 16

(e) 32

6. $\sqrt{\pi}\sqrt[3]{\pi} = \ldots$

(a) π^{-5}

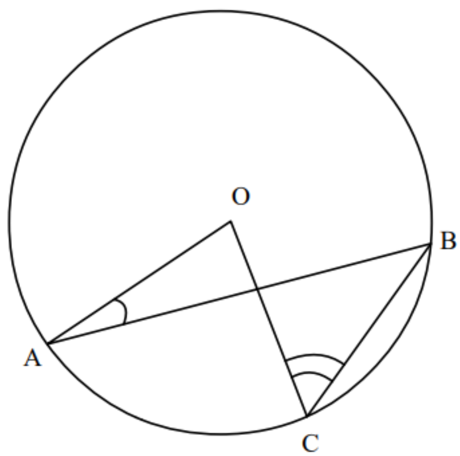
(b) π^5

(c) $\pi^{1/5}$

(d) $\pi^{1/6}$

(e) $\pi^{5/6}$

7. In the diagram, O is the center of the circle, $\angle OAB = 10°$ and $\angle OCB = 30°$. What is the measure of $\angle ABC$?



(a) $10°$

(b) $20°$

(c) $30°$

(d) $40°$

(e) $50°$

8. The following argument that $1 = 2$ contains a mathematical error. Which line (a,b,c,d, or e) **is not** a consequence of the previous line?

Let $a = b$.
(a) $ab = b^2$
(b) $ab - a^2 = b^2 - a^2$
(c) $a(b - a) = (b + a)(b - a)$
(d) $a = b + a$
(e) When $a = 1$, $b = 1$, so we get $1 = 1 + 1 = 2$.

9. The real numbers $x,\ y,$ and z satisfy $2^{x+y} = 10$, $2^{y+z} = 20$, and $2^{x+z} = 30$. Then 2^x is ...

(a) $\sqrt{15}$

(b) $\dfrac{\sqrt{6}}{2}$

(c) $\dfrac{3}{2}$

(d) $10\sqrt{6} - 20$

(e) None of the above

10. Two runners race on a circular track. The first runner completes laps every six minutes and the second runner completes laps every four minutes. If they start at the same point at the same time, how long will it take for the second runner to pass or 'lap' the first runner?

(a) 10 minutes

(b) 12 minutes

(c) 14 minutes

(d) 16 minutes

(e) None of the above

2023
Leap Frog Relay Grades 9-10

Solutions - Part I

1. The graphs of $y^6 = x^2$ and $|y| = |x|$ are plotted on the Cartesian plane. How many different regions do these graphs divide the plane into?

   (a) 8

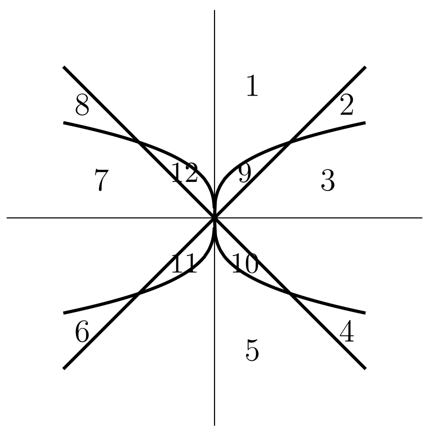
   (b) 11

   (c) 12

   (d) 13

   (e) None of these

   Solution. (c) Graphing both curves, we see that there are 12 regions.



2. What is the smallest positive whole number that can be divided by each of the whole numbers from 1 to 10 without any remainder?

   (a) 1260

   (b) 1680

   (c) 5040

   (d) 2520

   (e) None of these

   Solution. (d) If a number is divisible by 5, 7, 8, and 9, then it will be divisible by the numbers 1 to 10. The numbers 5, 7, 8, and 9 do not have any factors in common, so the answer will be the product of these four numbers (which is also the least common multiple):

$$5 \times 7 \times 8 \times 9 = 2520.$$

3. What is the last digit (ones place) in the sum $1^1 + 2^2 + 3^3 + ... + 10^{10}$?

(a) 7

(b) 1

(c) 9

(d) 4

(e) None of these

Solution. **(a)** The last digit is 7 since we can write down the first few terms as

$$1 + 4 + 27 + 256 + 5^5 + 6^6 + 7^7 + 8^8 + 9^9 + 10^{10}.$$

The last digit of the sum of the first five terms is 3 since 5^5 ends in 5. 6^6 will end in a 6. 7^2 ends in 9, so that means that 7^4 will end in 1, 7^6 will end in 9, and 7^7 will end in 3. Similarly, for 8^8, 8^2 ends in 4, 8^4 ends in 6, and 8^8 ends in 6. With a similar approach for 9, 9^9 ends in 9 and 10^{10} will end in 0. Adding these digits, we have

$$3 + 6 + 3 + 6 + 9 + 0 = 27$$

and the last digit will be 7.

4. An integer can be represented in binary form by writing it as a sum of powers of 2. For example,

$$\begin{aligned} 13 &= 8 + 4 + 1 \\ &= (a_4 \times 2^4) + (a_3 \times 2^3) + (a_2 \times 2^2) + (a_1 \times 2^1) + (a_0 \times 2^0). \end{aligned}$$

The coefficients next to the powers of two, 1's and 0's, are used to represent a number in binary. Hence $13 = (a_4a_3a_2a_1a_0)_2 = (10101)_2$.

What is 37 in binary?

(a) $(100001)_2$

(b) $(101011)_2$

(c) $(100101)_2$

(d) $(100010)_2$

(e) None of these

Solution. **(c)** We have

$$37 = 32 + 4 + 1 = 2^5 + 2^2 + 2^0 = (100101)_2.$$

5. How many different integers can be represented with a binary string of the form $(a_4a_3a_2a_1a_0)_2$?

(a) 15
(b) 64
(c) 31
(d) 16
(e) 32

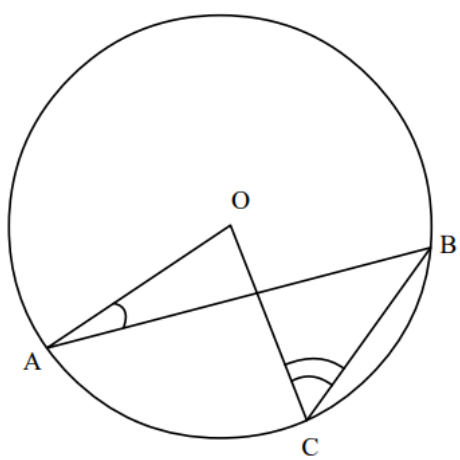
Solution. (e) Note that we can represent all the integers between $(00000)_2 = 0$ and $(11111)_2 = 2^5 - 1 = 31$, hence we have a total of 32 integers.

6. $\sqrt{\pi}\sqrt[3]{\pi} = \ldots$

(a) π^{-5}
(b) π^5
(c) $\pi^{1/5}$
(d) $\pi^{1/6}$
(e) $\pi^{5/6}$

Solution. (e) $\sqrt{\pi}\sqrt[3]{\pi} = \pi^{1/2}\pi^{1/3} = \pi^{1/2+1/3} = \pi^{5/6}$

7. In the diagram, O is the center of the circle, $\angle OAB = 10°$ and $\angle OCB = 30°$. What is the measure of $\angle ABC$?



(a) 10°
(b) 20°
(c) 30°
(d) 40°
(e) 50°

Solution. (b) Let $\angle ABC = x$. Then the arc measure of arc $\overset{\frown}{AC}$ is $2x$ since B is on the opposite side. Then $\angle AOC = 2x$. The two triangles share a pair of opposite angles which have the same measure which we will denote y. The sum of angles of a triangle is $180°$, so for the left triangle $y = 180° - \angle OAB - \angle AOC = 170° - 2x$ and for the right triangle $y = 180° - \angle OCB - \angle ABC = 150° - x$. Setting these equal, we have $170° - 2x = 150° - x$. Solving, we get $\angle ABC = x = 20°$.

8. The following argument that $1 = 2$ contains a mathematical error. Which line (a,b,c,d, or e) **is not** a consequence of the previous line?

   Let $a = b$.
   (a) $ab = b^2$
   (b) $ab - a^2 = b^2 - a^2$
   (c) $a(b - a) = (b + a)(b - a)$
   (d) $a = b + a$
   (e) When $a = 1$, $b = 1$, so we get $1 = 1 + 1 = 2$.

   Solution. (d) From line (c) to line (d), both sides are divided by $b - a$, which is zero since $a = b$. This is not a well-defined operation, so the statement in (d) is not a consequence of the previous line.

9. The real numbers $x,\ y,$ and z satisfy $2^{x+y} = 10$, $2^{y+z} = 20$, and $2^{x+z} = 30$. Then 2^x is ...

   (a) $\sqrt{15}$
   (b) $\frac{\sqrt{6}}{2}$
   (c) $\frac{3}{2}$
   (d) $10\sqrt{6} - 20$
   (e) None of the above

   Solution. (a) Since $20 = 2 \cdot 10$, we can use the first two equations to get $2^{y+z} = 2 \cdot 2^{x+y} = 2^{x+y+1}$. Then $y + z = x + y + 1$, so $z = x + 1$. Substituting this into the last equation we have

$$\begin{aligned} 2^{x+z} &= 2^{2x+1} = 30 \\ 2^{2x} \cdot 2 &= 2 \cdot 15 \\ 2^{2x} &= 15 \\ (2^x)^2 &= 15 \\ 2^x &= \sqrt{15}. \end{aligned}$$

10. Two runners race on a circular track. The first runner completes laps every six minutes and the second runner completes laps every four minutes. If they start at the same point at the same time, how long will it take for the second runner to pass or 'lap' the first runner?

    (a) 10 minutes
    (b) 12 minutes
    (c) 14 minutes
    (d) 16 minutes
    (e) None of the above

    Solution. (b) The speed of the first runner is 1 lap per 6 minutes and the speed of the second runner is 1 lap per 4 minutes. The difference in speed is $1/4 - 1/6 = 1/12$ laps per minute; that is, 1 lap per 12 minutes. So the second runner will lap the first in 12 minutes.

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Part II — Problems 11-20
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No calculators allowed

Correct Answer = 4, Incorrect Answer = −1, Blank = 0

11. How many solutions does the following equation have?

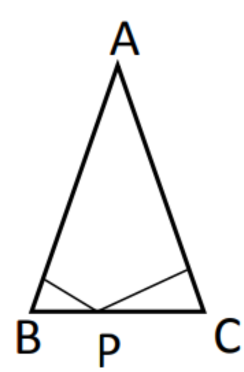
$$\left(\frac{2x^2-5}{3}\right)^{x^2-3x} = 1$$

(a) 2
(b) 3
(c) 4
(d) 5
(e) 6

12. Minnie the Miner mines 1200 pounds of ore that has an average of 3% gold in it and 2400 pounds of ore that is 6% gold. If 100 pounds of ore with 40% gold is removed from Minnie's stash, what is the percentage of gold in the remaining ore?

(a) 1%
(b) 2%
(c) 3%
(d) 4%
(e) 5%

13. Triangle ΔABC is isosceles with $\overline{AB} = \overline{AC}$ and $\overline{BC} = 65$ cm. Point P is a point on BC such that the perpendicular distances from P to $\overline{AB}$ and $\overline{AC}$ are 24 and 36 cm, respectively. What is the area of ΔABC in cm^2?



(a) 1254
(b) 1640
(c) 1950
(d) 2535
(e) 2942

14. Let $n = 1 \cdot 2 \cdot 3 \cdots 10$. How many distinct positive whole numbers are factors of n?

(a) 64

(b) 105

(c) 135

(d) 210

(e) 270

15. Compute the sum

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2022^2}+\frac{1}{2023^2}}.$$

(a) 2023

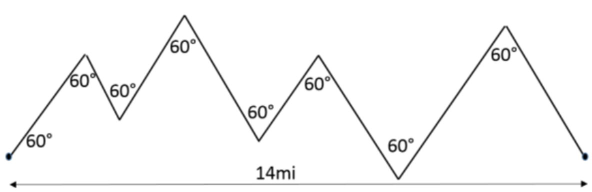
(b) $\dfrac{2022 \cdot 2024}{2023}$

(c) $\dfrac{2022^2}{2023}$

(d) $\dfrac{2021 \cdot 2023}{2024}$

(e) None of the above

16. The Pokemon Zig-a-Zag can only travel on zig-zagging roads. His friends made a road from his house to his practice field which is 14 miles away due East. The road starts out at a 60° angle from a horizontal line, and each zig forms a 60° angle with the zag before and after it, as shown in the figure below. Zig-a-Zag goes to the field and back each day on this road. How many miles does he travel on the road?



(a) 28

(b) 35

(c) 42

(d) 56

(e) Not enough information

17. A student kept rolling a fair die until the sum of her scores was a prime number. She had to roll three times. What is the probability that her last throw was a 6?

(a) 0

(b) 1/24

(c) 2/24

(d) 6/24

(e) 7/24

18. Determine the minimum possible value of

$$|x-1|+|x-2|+|x-3|+\cdots+|x-9|+|x-10|.$$

(a) 23
(b) 25
(c) 27
(d) 29
(e) 31

19. Let $f(x)=x^2+bx+c$ and $g(x)=x^2+dx+e$ where b, c, d, and e are real numbers with $b \neq d$. Given that $f(1)+f(10)+f(100)=g(1)+g(10)+g(100)$, what is the smallest positive value of x that will guarantee $f(x)$ equals $g(x)$?

(a) 1
(b) 11
(c) 37
(d) 74
(e) 111

20. Eleven non-negative numbers have a mean of 30. What is the largest possible value for the median of this collection?

(a) 22
(b) 33
(c) 44
(d) 55
(e) 66

2023
Leap Frog Relay Grades 9-10

Solutions - Part II

11. How many solutions does the following equation have?

$$\left(\frac{2x^2-5}{3}\right)^{x^2-3x} = 1$$

(a) 2

(b) 3

(c) 4

(d) 5

(e) 6

Solution. (e) There are three possibilities: (1) the base equals 1; (2) the exponent equals 0; or (3) the base equals -1 and the exponent is an even integer.
Case 1 has two solutions:

$$\begin{aligned}\frac{2x^2-5}{3} &= 1\\ 2x^2-5 &= 3\\ 2x^2 &= 8\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

Case 2 has two solutions:

$$\begin{aligned}x^2-3x &= 0\\ x(x-3) &= 0\\ x = 0,\ x &= 3\end{aligned}$$

Case 3 has two solutions:

$$\begin{aligned}\frac{2x^2-5}{3} &= -1\\ 2x^2-5 &= -3\\ 2x^2 &= 2\\ x^2 &= 1\\ x &= \pm 1\end{aligned}$$

in which case $x^2-3x = 1-3 = 2$ or $x^2-3x = 1+3 = 4$ which are both even. Since all these solutions are distinct, there are 6 total.

12. Minnie the Miner mines 1200 pounds of ore that has an average of 3% gold in it and 2400 pounds of ore that is 6% gold. If 100 pounds of ore with 40% gold is removed from Minnie's stash, what is the percentage of gold in the remaining ore?

(a) 1%

(b) 2%

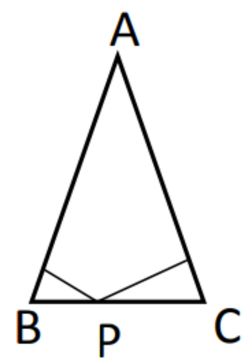
(c) 3%

(d) 4%

(e) 5%

Solution. **(d)** The total amount of gold is $1200 \cdot 0.03 + 2400 \cdot 0.06 = 36 + 144 = 180$ pounds. The removed gold is $100 \cdot 0.40 = 40$ pounds and the remaining ore is $3600 - 100 = 3500$ pounds. We have 140 pounds of gold in 3500 pounds of ore, so the percentage is $140/3500 = 0.04 = 4\%$.

13. Triangle ΔABC is isosceles with $\overline{AB} = \overline{AC}$ and $\overline{BC} = 65$ cm. Point P is a point on BC such that the perpendicular distances from P to $\overline{AB}$ and $\overline{AC}$ are 24 and 36 cm, respectively. What is the area of ΔABC in cm^2?



(a) 1254

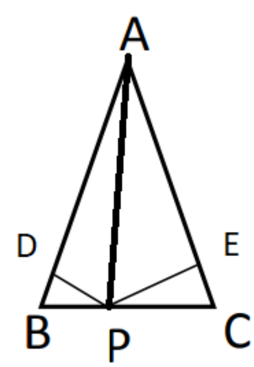
(b) 1640

(c) 1950

(d) 2535

(e) 2942

Solution. **(d)** Label the points D and E as below.



Since ΔABC is isosceles with $\overline{AB} = \overline{AC}$, the measures of $\angle B$ and $\angle C$ are equal. Then ΔPDB is similar to ΔPEC, so corresponding sides are proportional. We have $\overline{DP}/\overline{PE} = 24/36 = 2/3$, so $\overline{BP}/\overline{PC}$ is also equal to $2/3$. Then dividing up $\overline{BC} = 65$ into $2+3 = 5$ equal parts each of length 13, we have $\overline{BP} = 13 \cdot 2 = 26$ and $\overline{PC} = 13 \cdot 3 = 39$. Using the Pythagorean theorem on the right triangle ΔPDB, we have $\overline{DB} = \sqrt{26^2 - 24^2} = \sqrt{2^2(13^2 - 12^2)} = \sqrt{4 \cdot 25} = 10$. By proportionality $\overline{EC} = 15$. Now we can use the Pythagorean theorem on ΔAPD and ΔAPE to get

$$\overline{AP} = \sqrt{24^2 + \overline{AD}^2} = \sqrt{36^2 + \overline{AE}^2}.$$

Setting $\ell = \overline{AB} = \overline{AC}$, we have $\overline{AD} = \ell - 10$ and $\overline{AE} = \ell - 15$. Substituting into the equation above we have

$$\sqrt{24^2 + (\ell - 10)^2} = \sqrt{36^2 + (\ell - 15)^2}$$
$$24^2 + \ell^2 - 20\ell + 10^2 = 36^2 + \ell^2 - 30\ell + 15^2$$

So

$$\begin{aligned}10\ell &= 36^2 - 24^2 + 15^2 - 10^2 \\ &= (36+24)(36-24) + (15+10)(15-10) \\ &= 60 \cdot 12 + 25 \cdot 5 \\ &= 845.\end{aligned}$$

Finally, the area of ΔABC is the combined area of ΔABP and ΔACP which is

$$\frac{1}{2} \cdot 24 \cdot \ell + \frac{1}{2} \cdot 36 \cdot \ell = 30\ell = 3 \cdot 845 = 2535.$$

14. Let $n = 1 \cdot 2 \cdot 3 \cdots 10$. How many distinct positive whole numbers are factors of n?

(a) 64

(b) 105

(c) 135

(d) 210

(e) 270

Solution. (e) Let $x = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$. Then x has a prime factorization of $2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$. A factor of x can have 0, 1, 2, ..., 7, or 8 powers of 2. Similarly, any factor of x can have 0, 1, 2, 3, or 4 powers of 3; 0, 1, or 2 powers of 5, and 0 or 1 powers of 7. The total number of possibilities is then $(8+1)(4+1)(2+1)(1+1) = 270$.

15. Compute the sum

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\cdots+\sqrt{1+\frac{1}{2022^2}+\frac{1}{2023^2}}.$$

(a) 2023

(b) $\dfrac{2022\cdot 2024}{2023}$

(c) $\dfrac{2022^2}{2023}$

(d) $\dfrac{2021\cdot 2023}{2024}$

(e) None of the above

Solution. **(b)** The square roots have the form

$$\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}=\sqrt{\frac{n^2(n+1)^2+(n+1)^2+n^2}{n^2(n+1)^2}}=\sqrt{\frac{n^4+2n^3+3n^2+2n+1}{n^2(n+1)^2}}$$

$$=\sqrt{\frac{(n^2+n+1)^2}{n^2(n+1)^2}}=\frac{n^2+n+1}{n(n+1)}=\frac{n(n+1)+1}{n(n+1)}=1+\frac{1}{n(n+1)}=1+\frac{1}{n}-\frac{1}{n+1}.$$

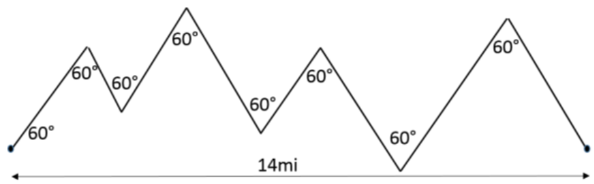
So the sum becomes

$$\left(1+\frac{1}{1}-\frac{1}{2}\right)+\left(1+\frac{1}{2}-\frac{1}{3}\right)+\left(1+\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(1+\frac{1}{2022}-\frac{1}{2023}\right).$$

The 1's will be added 2022 times and the fractions will cancel except for the first $1/1$ and the last $1/2023$. So the sum is

$$2022+1-\frac{1}{2023}=\frac{2023^2-1}{2023}=\frac{2022\cdot 2024}{2023}.$$

16. The Pokemon Zig-a-Zag can only travel on zig-zagging roads. His friends made a road from his house to his practice field which is 14 miles away due East. The road starts out at a 60° angle from a horizontal line, and each zig forms a 60° angle with the zag before and after it, as shown in the figure below. Zig-a-Zag goes to the field and back each day on this road. How many miles does he travel on the road?



(a) 28

(b) 35

(c) 42

(d) 56

(e) Not enough information

Solution. **(a)** If Zig-a-Zag traveled up and to the right until going halfway, then traveled down and to the right to the end point, it would form a equilateral triangle. In this case all three sides would be 14 miles, and so it would travel two of those sides to be 28 miles. Any other path that Zig-a-Zag takes, such as the one pictured above, can be broken down into its segments and reordered into that equilateral triangle.

17. A student kept rolling a fair die until the sum of her scores was a prime number. She had to roll three times. What is the probability that her last throw was a 6?

(a) 0

(b) 1/24

(c) 2/24

(d) 6/24

(e) 7/24

Solution. **(a)** There are four possible prime numbers to get as the sum of three rolls, namely 7, 11, 13, and 17 (note: the sums of 2, 3, and 5 would occur before three rolls). Given that the third roll was a 6, the sum 17 is not possible because the sum after the second roll would have been $17 - 6 = 11$ which is prime. Similarly, the sums 13 and 11 are not possible because the sum after two rolls would have been 7 and 5, respectively. And 7 cannot be obtained in three die rolls if the last roll is 6. Therefore the probability that her last roll was 6 is 0.

18. Determine the minimum possible value of

$$|x-1|+|x-2|+|x-3|+\cdots+|x-9|+|x-10|.$$

(a) 23

(b) 25

(c) 27

(d) 29

(e) 31

Solution. **(b)** We have $|x-a| = x-a$ when $x \geq a$ and $|x-a| = a-x$ when $x < a$. For example, if $x = 3$ then the expression above is equal to

$$2+1+0+1+2+3+4+5+6+7 = 31.$$

For whole numbers the minimum possible sum is

$$4+3+2+1+0+1+2+3+4+5 = 25$$

which occurs when $x = 5$ or $x = 6$. In fact, when x is any number between 5 and 6, we get

$$(x-1)+(x-2)+(x-3)+(x-4)+(x-5)+(6-x)+(7-x)+(8-x)+(9-x)+(10-x)$$
$$= 5x - 15 + 40 - 5x = 25.$$

So the minimum value is 25.

19. Let $f(x) = x^2 + bx + c$ and $g(x) = x^2 + dx + e$ where b, c, d, and e are real numbers with $b \neq d$. Given that $f(1) + f(10) + f(100) = g(1) + g(10) + g(100)$, what is the smallest positive value of x that will guarantee $f(x)$ equals $g(x)$?

(a) 1

(b) 11

(c) 37

(d) 74

(e) 111

Solution. **(c)** From $f(1) + f(10) + f(100) = g(1) + g(10) + g(100)$, we get

$$\begin{aligned} 1 + b + c + 100 + 10b + c + 10000 + 100b + c &= 1 + d + e + 100 + 10d + e + 10000 + 100d + e \\ 111b + 3c &= 111d + 3e \\ 111(b - d) &= 3(e - c). \qquad (1) \end{aligned}$$

Setting $f(x) = g(x)$, we get

$$\begin{aligned} x^2 + bx + c &= x^2 + dx + e \\ bx + c &= dx + e \\ (b - d)x &= e - c. \qquad (2) \end{aligned}$$

Combining (1) and (2), we get

$$\begin{aligned} 3(b - d)x = 3(e - c) &= 111(b - d) \\ (b - d)x &= (b - d)37. \end{aligned}$$

Since $b \neq d$ we have $b - d \neq 0$, so $x = 37$. In fact, this is the only solution.

20. Eleven non-negative numbers have a mean of 30. What is the largest possible value for the median of this collection?

(a) 22

(b) 33

(c) 44

(d) 55

(e) 66

Solution. **(d)** The sum of the eleven numbers must be $11 \cdot 30 = 330$. If we want the median to be positive, at most five of the numbers can be 0. Then the sixth number (in non-decreasing order) is the median. We can make the other six numbers equal or make one or more greater than the sixth number. Since these all must sum to 330, any increase to one number must make the sixth number smaller. Then the largest median will occur when all six numbers are equal, say to x, so $6x = 330$. It follows that the largest possible median is $x = 55$.