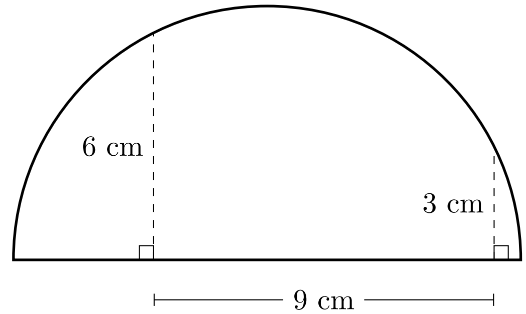
1

The figure below is not necessarily drawn to scale.



Find the area of the semicircle.

A. 45π cm^2

B. $\frac{45}{2}\pi$ cm^2

C. 81π cm^2

D. $\frac{81}{2}\pi$ cm^2

E. None of the above

2

A chessboard is an 8×8 rectangular grid. A rook is a chess piece that moves and attacks along the row or column in which it is located. In how many ways can four rooks be placed on a chessboard so that no rook is impeding the movement of another rook? (We are assuming that there are no other pieces on the board.) The figure below shows one such placement of the four rooks.

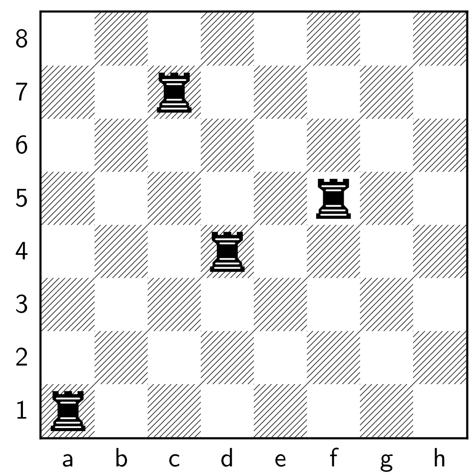
A. 19,600

B. 58,800

C. 75,264

D. 80,640

E. 117,600



3

The polynomial equation below has three real roots.

$$x^9 - 2023x^3 + \sqrt{2022} = 0$$

What is the product of its three real roots?

A. $-\sqrt[6]{2022}$
B. $-\sqrt[3]{2022}$
C. $-\sqrt{2022}$
D. $\sqrt[3]{2022}$
E. $\sqrt[6]{2022}$

4

Consider the following equation:

$$x^3 + x^2y - xy^2 - y^3 = 2023.$$

For each solution (x, y) for which x and y are positive whole numbers, let $P(x, y) = xy$. Find the sum of all such $P(x, y)$.

A. 30
B. 180
C. 60
D. 120
E. None of the above

5

A bicyclist goes up a hill at 30 km/hr and down the same hill at 90 km/hr. What is the cyclist's average speed for the trip?

A. 45 km/hr
B. 55 km/hr
C. 60 km/hr
D. 75 km/hr
E. None of the above

6

The number of pairs of integers (x, y) satisfying the equation $x^3 + y^3 = 6xy$ is ...

A. 1

B. 2

C. 6

D. infinitely many

E. None of the above

7

On average, how many times must a 6-sided die be rolled before the first occurrence of two 6's in a row?

A. 25

B. 36

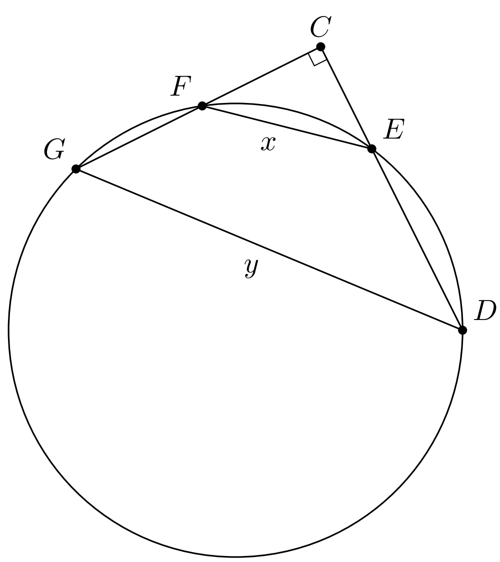
C. 42

D. 64

E. None of the above

8

In the figure below, $\angle DCG$ is a right angle. If $x = 12$ and $y = 16$, find the radius of the circle.



A. $\frac{10}{\sqrt{2}}$

B. 10

C. $\frac{20}{\sqrt{2}}$

D. 20

E. None of the above

9

Points A and B are on the parabola $y = 4x^2 + 7x - 1$, and the origin is the midpoint of AB. What is the length of AB

A. 7

B. $5 + \sqrt{2}$

C. $5 + \frac{1}{\sqrt{2}}$

D. $5\sqrt{2}$

E. $2\sqrt{5}$

10

A coin is altered so that the probability that it lands on heads is less than $1/2$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $1/6$. What is the probability that the coin lands on heads?

A. $\frac{\sqrt{2}-1}{2}$

B. $\frac{\sqrt{3}-1}{2}$

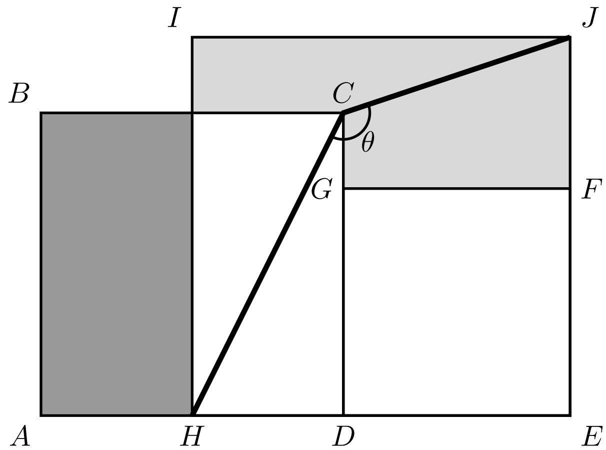
C. $\frac{\sqrt{15}-3}{6}$

D. $\frac{1}{4}$

E. $\frac{3-\sqrt{3}}{6}$

11

In the figure below, $ABCD$, $DEFG$, and $HIJE$ are all squares and the two shaded regions have equal areas. Find the measure of θ.



A. 120°

B. 135°

C. 145°

D. 150°

E. 160°

12

A sequence $\{a_k\}_{k=1}^{\infty}$ is defined as follows:

$$a_1 = m$$
$$a_2 = n$$
$$a_k = a_{k-2} + a_{k-1} \text{ for } k \geq 3$$

where $0 \leq m \leq n$. Consider all possible such sequences for which $a_7 = 2023$. If we compute the sum of the first 10 terms of each such sequence, how many distinct sums are possible?

A. 1

B. 2

C. 50

D. 51

E. Infinitely many

13

Consider the equation

$$3x + xy + y = 2020,$$

where x and y are positive integers. How many solutions does this equation have?

A. 0

B. 2

C. 3

D. 5

E. 10

14

Let S_N be the set of positive integers from 1 to N (inclusive) with $N \geq 2$. Consider the following 2-player game in which each player takes turns doing the following:

Choose a random number, k, from S_N.

- If $k = N$, then you win.
- If $k \neq N$, then remove all remaining numbers that are less than k and the next player takes their turn.

You and Mo have decided to play the game. One of you will choose the value of N and the other will choose who goes first. What should you do in order to maximize your probability of winning the game?

A. You choose N to be an even number.

B. You choose N to be an odd number.

C. You go first if Mo chooses N to be even and second if Mo chooses N to be odd.

D. You go first if Mo chooses N to be odd and second if Mo chooses N to be even.

E. It does not matter what you do.

15

The values of a, b, and c are defined as follows:

$$a = \log(9) \qquad b = 1 - \log(2) + \log(\sqrt{3}) \qquad c = \frac{1 + \log(8)}{2}$$

Arrange a, b, and c in ascending order.

A. $a < b < c$

B. $a < c < b$

C. $b < a < c$

D. $b < c < a$

E. None of the above

16

If x is a 2-digit positive whole number, what is the largest power of 3 in the prime factorization of $(x + 1)(x + 2)(x + 3) \cdots (x + 10)$?

A. 4

B. 5

C. 7

D. 8

E. 10

17

The number of pairs of integers (x, y) satisfying

$$x^2 + 3xy - 2y^2 = 122$$

is . . .

A. 0

B. 1

C. 2

D. 6

E. infinitely many

18

The number

$$40! = 40 \cdot 39 \cdot 38 \cdots 3 \cdot 2 \cdot 1$$

ends in how many zeros?

A. 5

B. 8

C. 9

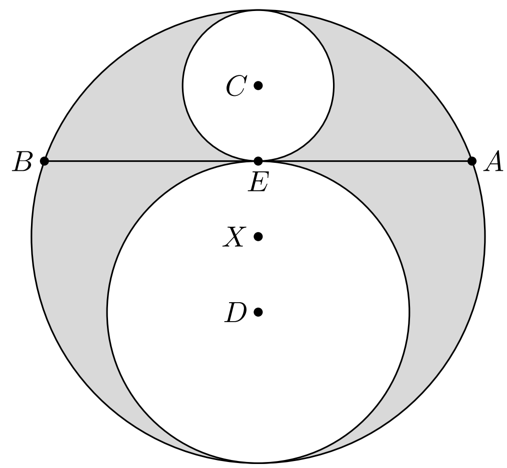
D. 10

E. None of the above

19

In the figure below,

- circle C and circle D are tangent to one another at E and are also tangent to circle X,
- C, D, E, and X are collinear, and
- $\overline{AB}$ is a chord of circle X that is tangent to circle C and circle D with length $AB = 8$ cm.



Find the area of the shaded region.

A. 4π cm^2

B. 8π cm^2

C. 12π cm^2

D. 16π cm^2

E. None of the above

20

A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top 1/8 of the volume of the mountain is above the water. What is the depth of the ocean at the base of the mountain in feet?

A. 4000

B. $2000(4 - \sqrt{2})$

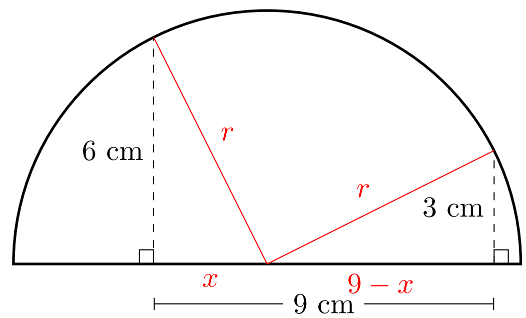
C. 6000

D. 7000

E. 64000

1

The figure below is not necessarily drawn to scale.



Find the area of the semicircle.

A. 45π cm^2

B. $\frac{45}{2}\pi$ cm^2

C. 81π cm^2

D. $\frac{81}{2}\pi$ cm^2

E. None of the above

Solution: The Pythagorean Theorem applied to the two right triangles gives us

$$\begin{aligned} r^2 = x^2 + 6^2 &= (9-x)^2 + 3^2 \\ x^2 + 36 &= 81 - 18x + x^2 + 9 \\ 18x &= 54 \\ x &= 3 \end{aligned}$$

So $r^2 = 45$, which gives the area of the semicircle to be $\dfrac{45}{2}\pi$ cm^2.

2

A chessboard is an 8×8 rectangular grid. A rook is a chess piece that moves and attacks along the row or column in which it is located. In how many ways can four rooks be placed on a chessboard so that no rook is impeding the movement of another rook? (We are assuming that there are no other pieces on the board.) The figure below shows one such placement of the four rooks.

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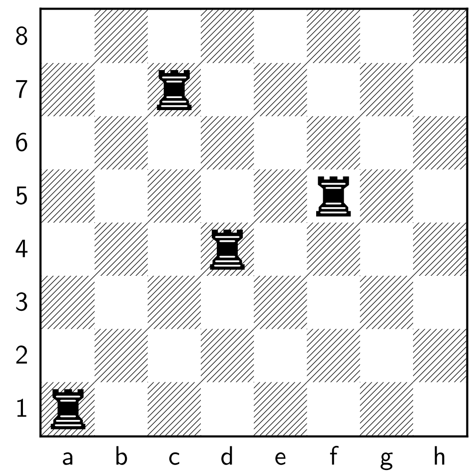


Solution: Each rook's position is specified by choosing a row and a column. To avoid impeding the movement of any other rook, each rook must be placed in a unique row and unique column. The number of ways we can choose four different rows is $\binom{8}{4}$. Similarly, the number of ways we can choose four different columns is $\binom{8}{4}$. Finally, each of the selected rows can be paired with any of the selected columns (without repeats). This can be done in 4! ways.

Applying the multiplication principle of counting yields

$$\binom{8}{4} \cdot \binom{8}{4} \cdot 4! = 70 \cdot 70 \cdot 24 = 117{,}600$$

possible ways to place the four rooks as required.

3

The polynomial equation below has three real roots.

$$x^9 - 2023x^3 + \sqrt{2022} = 0$$

What is the product of its three real roots?

A. $-\sqrt[6]{2022}$

B. $-\sqrt[3]{2022}$

C. $-\sqrt{2022}$

D. $\sqrt[3]{2022}$

E. $\sqrt[6]{2022}$

Solution: We'll consider two approaches.

1st Approach:

Let $y = \sqrt{2022}$, which gives us $y^2 + 1 = 2023$. This allows us to rewrite the equation as

$$x^9 - (y^2 + 1)x^3 + y = 0.$$

This equation can be rewritten as

$$x^3y^2 - y + x^3 - x^9 = 0,$$

which is quadratic in y. So solving for y yields

$$\begin{aligned} y &= \frac{1 \pm \sqrt{1 - 4(x^3)(x^3 - x^9)}}{2x^3} \\ &= \frac{1 \pm \sqrt{(2x^6 - 1)^2}}{2x^3}. \end{aligned}$$

This gives us two solutions for y:

$$y = \frac{1 + (2x^6 - 1)}{2x^3} = x^3 \tag{1}$$

and

$$y = \frac{1 - (2x^6 - 1)}{2x^3} = \frac{1 - x^6}{x^3}. \tag{2}$$

(1) gives us one real root of the original equation:

$$x_1 = \sqrt[3]{y} = \sqrt[6]{2022}. \tag{3}$$

(2) can be rewritten as

$$x^6 + yx^3 - 1 = 0. \tag{4}$$

(4) is quadratic in x^3 and can be solved to give us

$$x^3 = \frac{-y \pm \sqrt{y^2 + 4}}{2}, \tag{5}$$

which gives us the two remaining real roots of the original equation:

$$x_2 = \sqrt[3]{\frac{-y + \sqrt{y^2 + 4}}{2}} = \sqrt[3]{\frac{-\sqrt{2022} + \sqrt{2026}}{2}} \tag{6}$$

and

$$x_3 = \sqrt[3]{\frac{-y - \sqrt{y^2 + 4}}{2}} = \sqrt[3]{\frac{-\sqrt{2022} - \sqrt{2026}}{2}}. \tag{7}$$

So the product of the three real roots is

$$\begin{aligned} x_1 \cdot x_2 \cdot x_3 &= \sqrt[6]{2022} \cdot \sqrt[3]{\frac{-\sqrt{2022} + \sqrt{2026}}{2}} \cdot \sqrt[3]{\frac{-\sqrt{2022} - \sqrt{2026}}{2}} \\ &= \sqrt[6]{2022} \cdot \sqrt[3]{\frac{2022 - 2026}{4}} \\ &= \sqrt[6]{2022} \cdot \sqrt[3]{-1} \\ &= -\sqrt[6]{2022}. \end{aligned}$$

2nd Approach:

Make the substitution $y = x^3$ to give us

$$y^3 - 2023y + \sqrt{2022} = 0 \tag{8}$$

and notice that $y_1 = \sqrt{2022}$ is a solution to (8). This can be verified as shown below:

$$\begin{aligned}(\sqrt{2022})^3 - 2023(\sqrt{2022} + \sqrt{2022} &= 2022\sqrt{2022} - 2023\sqrt{2022} + \sqrt{2022}\\ &= 0\end{aligned}$$

This means that we can rewrite (8) as

$$(y - \sqrt{2022}) \cdot Q(y) = 0 \tag{9}$$

where $Q(y) = y^2 + \sqrt{2022}y - 1$ can be obtained by polynomial (or synthetic) division. The roots of $Q(y)$ are

$$y_2 = \frac{-\sqrt{2022} + \sqrt{2026}}{2} \tag{10}$$

and

$$y_3 = \frac{-\sqrt{2022} - \sqrt{2026}}{2} \tag{11}$$

Using $y = x^3$, we get the three real roots of the original equation are

$$\begin{aligned}x_1 &= \sqrt[6]{2022}\\ x_2 &= \sqrt[3]{\frac{-\sqrt{2022} + \sqrt{2026}}{2}}\\ x_3 &= \sqrt[3]{\frac{-\sqrt{2022} - \sqrt{2026}}{2}}\end{aligned}$$

which are the same as what we found in our 1st approach above.

$\boxed{4}$

Consider the following equation:

$$x^3 + x^2y - xy^2 - y^3 = 2023.$$

For each solution (x, y) for which x and y are positive whole numbers, let $P(x, y) = xy$. Find the sum of all such $P(x, y)$.

A. 30

B. 180

C. 60

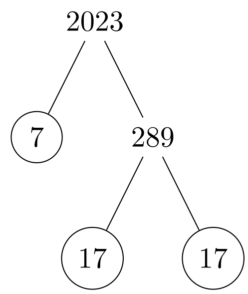
D. 120

E. None of the above

Solution: We begin by factoring the left-hand side of the equation to get

$$\begin{aligned} x^3 + x^2y - xy^2 - y^3 &= 2023 \\ x^2(x+y) - y^2(x+y) &= 2023 \\ (x+y)(x^2-y^2) &= 2023 \\ (x+y)(x+y)(x-y) &= 2023 \\ (x-y)(x+y)^2 &= 2023 \end{aligned}$$

This suggests trying to factor 2023 into three factors, two of which are the same.



We see that the prime factorization of 2023 is $7 \cdot 17^2$. This means that

$$\begin{cases} x - y = 7 \\ x + y = 17 \end{cases}$$

Solving this system yields $x = 12$ and $y = 5$ as the only solution. So the sum we are being asked for consists of a single term:

$$\sum P(x,y) = 12 \cdot 5 = 60.$$

5

A bicyclist goes up a hill at 30 km/hr and down the same hill at 90 km/hr. What is the cyclist's average speed for the trip?

A. 45 **km/hr**

B. 55 km/hr

C. 60 km/hr

D. 75 km/hr

E. None of the above

Solution: We can assume the length of the hill is 90 km. The trip takes 3 hours up the hill and 1 hour down. So the average speed is $180/4 = 45$ km/hr.

6

The number of pairs of integers (x, y) satisfying the equation $x^3 + y^3 = 6xy$ is ...

A. 1

B. 2

C. 6

D. infinitely many

E. None of the above

Solution: We apply the formula

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$$

to conclude

$$8 = x^3 + y^3 - 6xy + 8 = (x + y + 2)(x^2 + y^2 + 4 - xy - 2x - 2y).$$

which implies $x + y + 2$ is a factor of 8. If $x + y + 2 = 8$, then the equation

$$36 - 12 + 4 - 3xy = 1$$

implies that $xy = 9$ and we obtain the solution $(x, y) = (3, 3)$. Similarly, the case $x + y + 2 = 2$ produces $(x, y) = (0, 0)$. Following similar computations, we can check that other cases fail to give solutions.

7

On average, how many times must a 6-sided die be rolled before the first occurrence of two 6's in a row?

A. 25

B. 36

C. 42

D. 64

E. None of the above

Solution: Let E_1 denote the expected number of rolls needed to get the first 6. Then

$$E_1 = \frac{1}{6}(1) + \frac{5}{6}(1 + E_1),$$

which we can solve to get $E_1 = 6$.

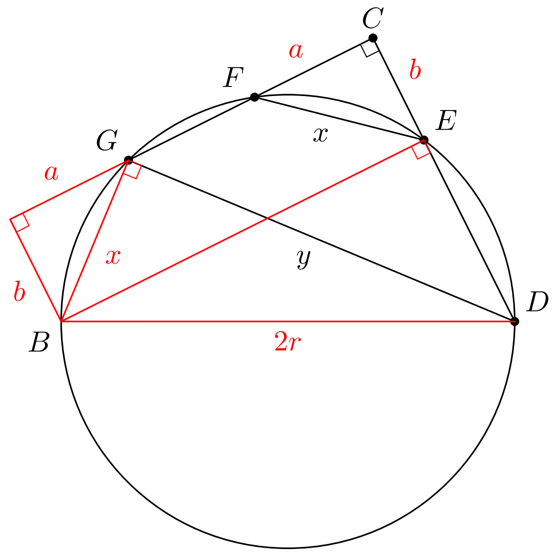
Now, let E_2 denote the expected number of rolls needed to get the second 6 in a row. Then

$$E_2 = E_1 + \frac{1}{6}(1) + \frac{5}{6}(1 + E_2),$$

which we can solve to get $E_2 = 42$.

8

In the figure below, $\angle DCG$ is a right angle. If $x = 12$ and $y = 16$, find the radius of the circle.



A. $\frac{10}{\sqrt{2}}$

B. 10

C. $\frac{20}{\sqrt{2}}$

D. 20

E. None of the above

Solution: From the figure, we obtain the relation

$$(2r)^2 = x^2 + y^2,$$

from which we get $r = 10$.

9

Points A and B are on the parabola $y = 4x^2 + 7x - 1$, and the origin is the midpoint of AB. What is the length of AB

A. 7

B. $5 + \sqrt{2}$

C. $5 + \frac{1}{\sqrt{2}}$

D. $5\sqrt{2}$

E. $2\sqrt{5}$

Solution: If (x, y) is the coordinate of A, then that of B is $(-x, -y)$. The equation $-4x^2 - 7x + 1 = 4(-x)^2 + 7(-x) - 1$ simplifies to $8x^2 - 2 = 0$ from which we deduce that $x = \pm 1/2$. We obtain the pair of points $(1/2, 7/2)$ and $(-1/2, -7/2)$ whose distance is $\sqrt{1^2 + 7^2} = 5\sqrt{2}$.

10

A coin is altered so that the probability that it lands on heads is less than $1/2$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $1/6$. What is the probability that the coin lands on heads?

A. $\frac{\sqrt{2}-1}{2}$

B. $\frac{\sqrt{3}-1}{2}$

C. $\frac{\sqrt{15}-3}{6}$

D. $\frac{1}{4}$

E. $\frac{3-\sqrt{3}}{6}$

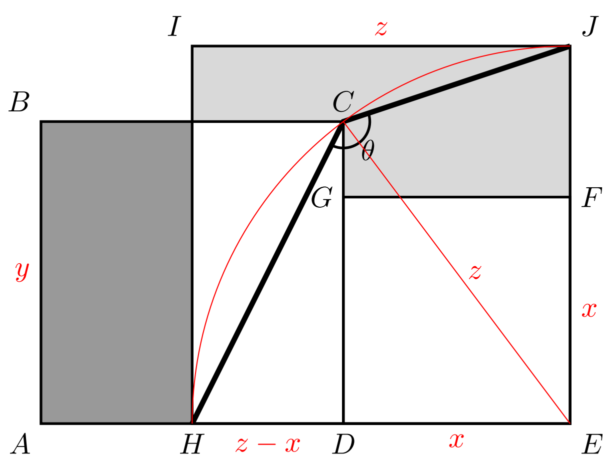
Solution: If x is the probability of flipping heads, then

$$\binom{4}{2}x^2(1-x)^2 = \frac{1}{6}.$$

from which we deduce that $x(1-x) = \frac{1}{6}$ since both x and $1-x$ are nonnegative. Quadratic formula gives two solutions $\frac{3\pm\sqrt{3}}{6}$. The solution less than $1/2$ is $\frac{3-\sqrt{3}}{6}$.

11

In the figure below, $ABCD$, $DEFG$, and $HIJE$ are all squares and the two shaded regions have equal areas. Find the measure of θ.



A. 120°

B. 135°

C. 145°

D. 150°

E. 160°

Solution: The area of the rectangular shaded region is

$$\begin{aligned} A_1 &= [y - (z - x)] \cdot y \\ &= y^2 - yz + xy \end{aligned}$$

This means that the area of the other shaded region is also $y^2 - yz + xy$. Looking at square $HIJE$, we have

$$\begin{aligned} \text{Area of } HIJE = z^2 &= y \cdot (z - x) + x^2 + (y^2 - yz + xy) \\ &= yz - xy + x^2 + y^2 - yz + xy \\ &= x^2 + y^2 \end{aligned}$$

This means that $\triangle EDC$ is a right triangle with hypotenuse z. So we can consider the circle centered at E with radius z. Since the minor arc HJ has measure $90°$, the major arc HJ must have measure $270°$. Since this major arc is subtended by the inscribed angle θ, it follows that $\theta = 135°$.

12

A sequence $\{a_k\}_{k=1}^{\infty}$ is defined as follows:

$$\begin{aligned} a_1 &= m \\ a_2 &= n \\ a_k &= a_{k-2} + a_{k-1} \text{ for } k \geq 3 \end{aligned}$$

where $0 \leq m \leq n$. Consider all possible such sequences for which $a_7 = 2023$. If we compute the sum of the first 10 terms of each such sequence, how many distinct sums are possible?

A. 1

B. 2

C. 50

D. 51

E. Infinitely many

Solution: We can write out the first 10 terms of such a sequence as follows:

$$\begin{aligned} a_1 &= m \\ a_2 &= n \\ a_3 &= a_1 + a_2 \\ &= m + n \\ a_4 &= a_2 + a_3 \\ &= m + 2n \\ a_5 &= a_3 + a_4 \\ &= 2m + 3n \\ a_6 &= a_4 + a_5 \\ &= 3m + 5n \end{aligned}$$

$$\begin{aligned} a_7 &= a_5 + a_6 \\ &= 5m + 8n \\ a_8 &= a_6 + a_7 \\ &= 8m + 13n \\ a_9 &= a_7 + a_8 \\ &= 13m + 21n \\ a_{10} &= a_8 + a_9 \\ &= 21m + 34n \end{aligned}$$

Adding the first 10 terms gives us

$$a_1 + \cdots + a_{10} = 55m + 88n.$$

Notice that this is exactly 11 times a_7. So

$$a_1 + \cdots + a_{10} = 11a_7 = 11 \cdot 2023.$$

13

Consider the equation

$$3x + xy + y = 2020,$$

where x and y are positive integers. How many solutions does this equation have?

A. 0

B. 2

C. 3

D. 5

E. 10

Solution: We can rewrite the equation by factoring x from the first two terms on the left-hand side to get

$$x(3+y) + y = 2020.$$

Now, adding 3 to both sides gives us

$$\begin{aligned} x(3+y) + 3 + y &= 2023 \\ (y+3)(x+1) &= 2023. \end{aligned}$$

Since x and y are positive integers, it follows that $y+3$ and $x+1$ must be integer factors of 2023. Since the prime factorization of 2023 is $7 \cdot 17^2$, we have the following possible ways to write 2023 as a product of two integers:

$(y+3)(x+1)$		y	x
1	2023	-2	2022
7	289	4	288
119	17	116	16
289	7	286	6
2023	1	2022	-2

$(y+3)(x+1)$		y	x
-1	-2023	-4	-2024
-7	-289	-10	-290
-119	-17	-122	-18
-289	-7	-292	-8
-2023	-1	-2026	-2

Since x and y must be positive integers, there are only three possible solutions: $(288, 4)$, $(16, 116)$, and $(6, 286)$.

14

Let S_N be the set of positive integers from 1 to N (inclusive) with $N \geq 2$. Consider the following 2-player game in which each player takes turns doing the following:

Choose a random number, k, from S_N.

- If $k = N$, then you win.
- If $k \neq N$, then remove all remaining numbers that are less than k and the next player takes their turn.

You and Mo have decided to play the game. One of you will choose the value of N and the other will choose who goes first. What should you do in order to maximize your probability of winning the game?

A. You choose N to be an even number.

B. You choose N to be an odd number.

C. You go first if Mo chooses N to be even and second if Mo chooses N to be odd.

D. You go first if Mo chooses N to be odd and second if Mo chooses N to be even.

E. It does not matter what you do.

Solution: If we record what each player draws on their turn, the result will be a strictly increasing subsequence of $\{1, 2, \ldots, N\}$ ending in N. Note the following:

1. Each of these subsequences can be constructed by taking a subset of $\{1, 2, \ldots, N-1\}$ and appending N to it. The number of subsets of $\{1, 2, \ldots, N-1\}$ is 2^{N-1}. So there are a total of 2^{N-1} ways in which this game can play out.
2. If the length of the subsequence is odd, then Player 1 wins. Otherwise, Player 2 wins.

We can count the number of subsequences of length k by noting that we need to choose $k-1$ of the numbers from 1 to $N-1$ for the subsequence.

$$\underbrace{1,\ 2,\ 3,\ \ldots,\ N-2,\ N-1,}_{\text{Want to keep } k-1 \text{ of these.}}\ N$$

So the number of subsequences of length k is just $\binom{N-1}{N-k}$, where k is any whole number from 1 to N. In other words, the the number of subsequences of length k is the $(k-1)^{\text{th}}$ binomial coefficient in the expansion of $(1+x)^{N-k}$:

$$(1+x)^{N-k} = \binom{N-1}{0} + \binom{N-1}{1}x + \binom{N-1}{2}x^2 + \cdots + \binom{N-1}{N-2}x^{N-2} + \binom{N-1}{N-1}x^{N-1}$$

By substituting $x = -1$, we get

$$\begin{aligned} 0 &= \binom{N-1}{0} - \binom{N-1}{1} + \binom{N-1}{2} - \cdots + \binom{N-1}{N-2}(-1)^{N-2} + \binom{N-1}{N-1}(-1)^{N-1} \\ &= \sum_{\substack{j \text{ even} \\ 0 \leq j \leq N-1}} \binom{N-1}{j} - \sum_{\substack{j \text{ odd} \\ 0 \leq j \leq N-1}} \binom{N-1}{j} \end{aligned}$$

So the total number of subsequences of odd length, $\sum_{\substack{j \text{ even} \\ 0 \le j \le N-1}} \binom{N-1}{j}$, is equal to the total number of subsequences of even length, $\sum_{\substack{j \text{ odd} \\ 0 \le j \le N-1}} \binom{N-1}{j}$.

Therefore, regardless of the value of N, each player has a 50% chance of winning.

15

The values of a, b, and c are defined as follows:

$$a = \log(9) \qquad b = 1 - \log(2) + \log(\sqrt{3}) \qquad c = \frac{1 + \log(8)}{2}$$

Arrange a, b, and c in ascending order.

A. $a < b < c$

B. $a < c < b$

C. $b < a < c$

D. $b < c < a$

E. None of the above

Solution: First, we can rewrite b as follows:

$$b = 1 - \log(2) + \frac{1}{2}\log(3)$$

We will multiply each expression by 2 and compare $2a$, $2b$, and $2c$.

$$\begin{aligned} 2a &= 2\log(9) \\ &= \log(81) \end{aligned}$$

$$\begin{aligned} 2b &= 2\log(10) - 2\log(2) + \log(3) \\ &= \log(100) - \log(4) + \log(3) \\ &= \log(25) + \log(3) \\ &= \log(75) \end{aligned}$$

$$\begin{aligned} 2c &= 1 + \log(8) \\ &= \log(10) + \log(8) \\ &= \log(80) \end{aligned}$$

Since log is an increasing function, it follows that $2b < 2c < 2a$. So $b < c < a$.

16

If x is a 2-digit positive whole number, what is the largest power of 3 in the prime factorization of $(x+1)(x+2)(x+3)\cdots(x+10)$?

A. 4

B. 5

C. 7

D. 8

E. 10

Solution: The number $W = (x+1)(x+2)(x+3)\cdots(x+10)$ is the product of 10 consecutive whole numbers. Thus the largest number of 3's in the factorization of W will occur when $x+1$ is divisible by 3, in which case $x+4$, $x+7$, and $x+10$ are all divisible by 3. Notice also that at most one of these numbers is divisible by 9 unless $x+1$ is divisible by 9, in which case $x+10$ is also. The 2-digit number with the largest number of factors of 3 is $81 = 3^4$, so we can let $x+1 = 81$ or $x+10 = 81$. Either way, we obtain $4+1+1+2 = 8$ factors of 3 in the prime factorization of W.

17

The number of pairs of integers (x, y) satisfying

$$x^2 + 3xy - 2y^2 = 122$$

is ...

A. 0

B. 1

C. 2

D. 6

E. infinitely many

Solution: The equation is equivalent to

$$(2x+3y)^2 - 17y^2 = 488.$$

This implies that in modulo 17, 488 is a square. Note that $488 \equiv 12 \pmod{17}$ and we can check 12 is not a square in modulo 17.

18

The number

$$40! = 40 \cdot 39 \cdot 38 \cdots 3 \cdot 2 \cdot 1$$

ends in how many zeros?

A. 5

B. 8

C. 9

D. 10

E. None of the above

Solution: Since there are more factors 2 than 5, the number of 0's is determined by the largest exponent of 5 that is a factor of 40!. This number is

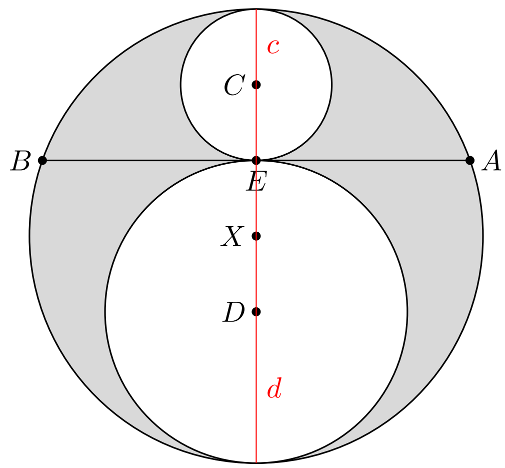
$$\left\lfloor \frac{40}{5} \right\rfloor + \left\lfloor \frac{40}{25} \right\rfloor = 8 + 1 = 9,$$

where $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

19

In the figure below,

- circle C and circle D are tangent to one another at E and are also tangent to circle X,
- C, D, E, and X are collinear, and
- $\overline{AB}$ is a chord of circle X that is tangent to circle C and circle D with length $AB = 8$ cm.



Find the area of the shaded region.

A. 4π cm^2

B. 8π cm^2

C. 12π cm^2

D. 16π cm^2

E. None of the above

Solution: Let c and d denote the radii of circle C and circle D and let r denote the radius of circle X. Then the area of the shaded region is

$$A = \pi r^2 - \pi c^2 - \pi d^2.$$

Since $r = c + d$, it follows that

$$\begin{aligned} A &= \pi(c+d)^2 - \pi c^2 - \pi d^2 \\ &= 2\pi cd. \end{aligned}$$

The chord-chord theorem gives us

$$\begin{aligned}(2c)(2d) &= 4^2\\ 4cd &= 16\\ 2\pi cd &= 8\pi.\end{aligned}$$

20

A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top 1/8 of the volume of the mountain is above the water. What is the depth of the ocean at the base of the mountain in feet?

A. 4000

B. $2000(4-\sqrt{2})$

C. 6000

D. 7000

E. 64000

Solution: The height of the small cone above the water is $\sqrt[3]{1/8} = 1/2$ that of the mountain. Thus the depth of the water is $8000 - 4000 = 4000$.