CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART I

April 15, 2023

Prepare your Scantron form:

- **Name:** Your full name
- **Subject:** Your school name
- **Test No:** Mad Hatter 11-12
- **Period:** 1

Rules

- Please turn off your cell phones.
- No calculators or any other devices are allowed.
- You may use scratch paper.
- There will be two parts, 30 problems each.
- You will have 2 minutes to solve each problem.
- Record your answer on the Scantron form during those 2 minutes. Problems will not be shown again.
- Each correct answer is worth 1 point. Blank or incorrect is 0 points.
- Ties, if any, will be broken at the awards ceremony.
- Let the fun begin!

1. If a, b, and c are real numbers such that $a + b = 25$, $b + c = 43$, and $c + a = 28$, what is the value of abc?

(a) 1200

(b) 1800

(c) 2300

(d) 4600

(e) None of the above

2. Misha has five cats and three dogs. The average weight of the cats is 9 lbs. and the average weight of the dogs is 25 lbs. What is the average weight of all eight of Misha's pets?

(a) 14 lbs

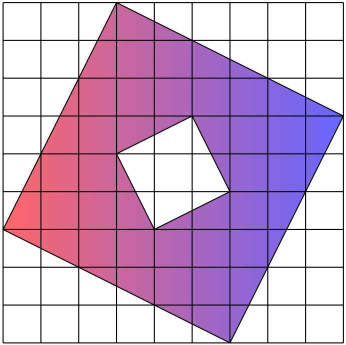
(b) 15 lbs

(c) 16 lbs

(d) 17 lbs

(e) None of the above

3. The side length of each small square in the grid below is 1 unit. Find the area of the shaded region.



(a) 32 square units

(b) 35 square units

(c) 36 square units

(d) 40 square units

(e) 42 square units

4. Compute:

$$3 + 13 + 23 + \cdots + 2013 + 2023 = \ldots$$

(a) 4,075

(b) 204,626

(c) 205,639

(d) 409,252

(e) 411,278

5. Points A, B, and C lie on a straight line. Segment AB is a side of a regular hexagon while BC is a side of a regular octagon as shown in the picture below. What is the degree measure of the angle marked in the picture?

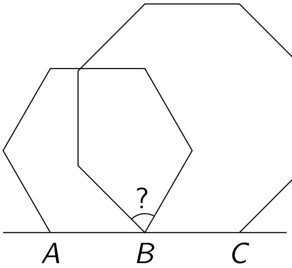
(a) 72

(b) 75

(c) 80

(d) 90

(e) None of the above



6. If x and y are real numbers such that $x + y = 10$, what is the largest possible value of xy?

(a) 25

(b) 50

(c) 75

(d) 100

(e) None of the above

7. Abe and Bob were solving a quadratic equation $x^2 + ax + b = 0$. Abe mis-copied the coefficient a but did everything else correctly and obtained the roots 8 and -1. Bob mis-copied the coefficient b but did everything else correctly and obtained the roots 3 and -5. What are the roots of the original equation?

(a) 10 and -3

(b) 4 and -2

(c) 4 and -6

(d) 2 and -4

(e) None of the above

8. What is the value of $\cos 15^\circ$?

(a) $\dfrac{\sqrt{6}-\sqrt{2}}{2}$

(b) $\dfrac{\sqrt{3}-\sqrt{2}}{2}$

(c) $\dfrac{\sqrt{2}+1}{2}$

(d) $\dfrac{\sqrt{3}+1}{4}$

(e) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

9. How many ways are there to select five cells in a 5×5 table so that all of the selected cells are in different rows and in different columns?

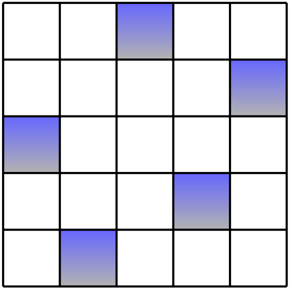
(a) 5

(b) 15

(c) 25

(d) 120

(e) 3125



10. For what value of c is the parabola $y = x^2 + 2x + c$ tangent to the line $y = 3$?

(a) 3

(b) 4

(c) 5

(d) 6

(e) None of the above

11. Ann writes out all factors of 1,000,000. Ben randomly picks one of them. What is the probability that the factor Ben picks is odd?

(a) $\frac{1}{7}$

(b) $\frac{1}{2}$

(c) $\frac{5}{6}$

(d) $\frac{6}{7}$

(e) 1

12. A point with coordinates (a, b) where $a, b \neq 0$, is reflected about the x-axis, then its image is reflected about the line $y = x$, and finally, that image is reflected about the y-axis. What are the coordinates of the result?

(a) (a, b)

(b) (b, a)

(c) $(a, -b)$

(d) $(b, -a)$

(e) $(-b, -a)$

13. If $f(x) = x^3 + x + 1$ and f^{-1} is its inverse function, then what is $f^{-1}(31)$?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

14. Let S be the number of perfect squares between 1 and 2023^6 (inclusive) and let C be the number of perfect cubes between 1 and 2023^6 (again, inclusive). Which of the following is true about S and C?

(a) $S = 2023C$

(b) $S^3 = C^2$

(c) $S = C^{2023}$

(d) $S = 2023^C$

(e) $S^2 = 2023C^3$

15. Four circles of radius 1, each of which is tangent to two others, are inscribed in a larger circle as shown. What is the radius of the large circle?

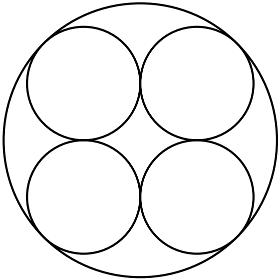
(a) 2

(b) 3

(c) $2\sqrt{2}$

(d) $2 + \frac{1}{\sqrt{2}}$

(e) $\sqrt{2} + 1$



16. Let $f(x) = x + 123$, $g(x) = x - 456$, and $h(x) = 789x$. Which of the following has the largest value?

(a) $f(g(h(2023)))$

(b) $f(h(g(2023)))$

(c) $g(f(h(2023)))$

(d) $g(h(f(2023)))$

(e) $h(f(g(2023)))$

17. How many subsets of $\{a, b, c, d, e, f, g, h\}$ contain both a and b but do not contain c?

(a) 5

(b) 25

(c) 32

(d) 128

(e) 256

18. What is the smallest natural number n such that the number $2023n$ is a perfect square?

(a) 2

(b) 3

(c) 5

(d) 7

(e) 11

19. How many values $0 \leq x \leq 20$ satisfy $\sin(x) = \cos(x)$?

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

20. For how many values of the digit D is the number 2023D2023 divisible by 33?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 10

21. What is the smallest possible positive value of

$$1 * 2 * 3 * \ldots * 2021 * 2022 * 2023$$

where we replace each $*$ by either $+$ or $-$?

(a) 1

(b) 2

(c) 1011

(d) 1012

(e) None of the above

22. Point A has coordinates $(3, 3)$; point B has coordinates $(5, 8)$; point O is the origin. What is the value of $\tan(\angle AOB)$?

(a) $\frac{3}{5}$

(b) $\frac{3}{13}$

(c) $\frac{2}{11}$

(d) $\frac{5}{16}$

(e) None of the above

23. Matt drilled a cylindrical hole of radius 1 in a 3 in $\times$ 3 in $\times$ 3 in cube. What is the difference between the surface area of the obtained solid and that of the original cube?

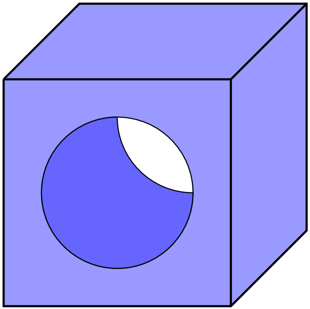
(a) $2\pi + 3$ in^2

(b) $6 - 2\pi$ in^2

(c) 4π in^2

(d) $6\pi - 2$ in^2

(e) 6π in^2



24. If a, b, and c are positive integers such that

$$12^a \cdot 18^b = 6^c,$$

what is the value of $\dfrac{a+b}{c}$?

(a) $\dfrac{2}{3}$

(b) $\dfrac{3}{4}$

(c) 1

(d) $\dfrac{4}{3}$

(e) $\dfrac{3}{2}$

25. What is the remainder when the number 2023^{2023} is divided by 5?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

26. The math club at Circle High ordered some apple pies and cherry pies to celebrate the Pi Day. There were twice as many apple pies as cherry pies ordered. The bakery accidentally switched the prices of the pies which increased the bill by 20%. What is the ratio of the price of the apple pie to that of the cherry pie?

(a) 3 : 5

(b) 4 : 7

(c) 5 : 6

(d) 4 : 3

(e) 7 : 6

27. Find the coefficient of x^{46} in the expansion of $(2x^3 - x^5)^{10}$.

(a) 4

(b) 45

(c) 180

(d) 1,024

(e) 11,520

28. In the first row of the table shown below, Tom writes numbers a and b such that $a > b$. In the second row he writes their sum on the left and their difference on the right. He then continues filling the table so that in each row, the left number is the sum of the two numbers in the previous row and the right number is the difference of the two numbers in the previous row. Two numbers in the table are shown. What is the number a?

a	b
	8
40	

(a) 7

(b) 8

(c) 9

(d) 10

(e) 11

29. Integers a, b, c, and d, not necessarily distinct, are chosen independently and at random from 0 to 2023, inclusive. What is the probability that $ab + cd$ is even?

(a) $\frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{3}{4}$

(d) $\frac{3}{8}$

(e) $\frac{5}{8}$

30. If the polynomial $P(x) = ax^5 + 2x^4 + bx^3 - x^2 + 2x + 3$ is divisible by $x + 1$, what is the value of $a + b$?

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

End of Part I

- Please remain seated until all Scantron forms have been collected.
- We will take a 10-minute break, then do Part II.

Answers

1. c
2. b
3. d
4. c
5. b
6. a
7. d
8. e
9. d
10. b
11. a
12. b
13. d
14. a
15. e

16. d
17. c
18. d
19. e
20. a
21. b
22. b
23. c
24. a
25. c
26. b
27. c
28. a
29. e
30. e

CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART II

April 15, 2023

Prepare your Scantron form:

- **Name:** Your full name
- **Subject:** Your school name
- **Test No:** Mad Hatter 11-12
- **Period:** 2

Rules

- Please turn off your cell phones.
- No calculators or any other devices are allowed.
- You may use scratch paper.
- Part II contains 30 problems.
- You will have 2 minutes to solve each problem.
- Record your answer on the Scantron form during those 2 minutes. Problems will not be shown again.
- Each correct answer is worth 1 point. Blank or incorrect is 0 points.
- Ties, if any, will be broken at the awards ceremony.
- Let the fun begin!

1. A rhombus has four sides of length a and two diagonals of length b and c. Which of the following must be true?

   (a) $a^2 = b^2 + c^2$

   (b) $a^2 + b^2 = c^2$

   (c) $2a^2 = b^2 + c^2$

   (d) $a^2 + b^2 = 2c^2$

   (e) $4a^2 = b^2 + c^2$

2. Evaluate:

$$\frac{\ln^2 9 - \ln^2 4}{\ln^2 3 - \ln^2 2}.$$

(a) 2

(b) 4

(c) 6

(d) 9

(e) 36

3. How many real-number solutions does the equation $x^6 + 12x^3 + 32 = 0$ have?

(a) 0

(b) 1

(c) 2

(d) 6

(e) Infinitely many

4. When 2023 is written in base 4, how many digits does it have?

(a) 4
(b) 5
(c) 6
(d) 7
(e) 8

5. Find a complex number z such that

$$(1 + 2\mathrm{i}) + (3 + 4\mathrm{i})z = 5 + 6\mathrm{i}.$$

(a) $\frac{4}{3} + \mathrm{i}$

(b) $1 + \frac{4}{3}\mathrm{i}$

(c) $\frac{7}{5} - \frac{4}{5}\mathrm{i}$

(d) $\frac{28}{25} - \frac{4}{25}\mathrm{i}$

(e) $\frac{4}{25} + \frac{28}{25}\mathrm{i}$

6. If x is a real number, $\lfloor x \rfloor$ is the integer part of x, that is, the largest integer that does not exceed x. Of the following, which represents the smallest number?

(a) $\pi - \lfloor \pi \rfloor$

(b) $2\pi - \lfloor 2\pi \rfloor$

(c) $4\pi - \lfloor 4\pi \rfloor$

(d) $8\pi - \lfloor 8\pi \rfloor$

(e) $16\pi - \lfloor 16\pi \rfloor$

7. Nathan draws four squares. The first one has a side length of 2 cm. The second one has a side length of 4 cm and a vertex placed in the center of the first square. The third one has a side length of 6 cm and a vertex placed in the center of the second square. The last one has a side length of 8 cm and a vertex placed in the center of the third square. What is the area of the entire shaded figure?

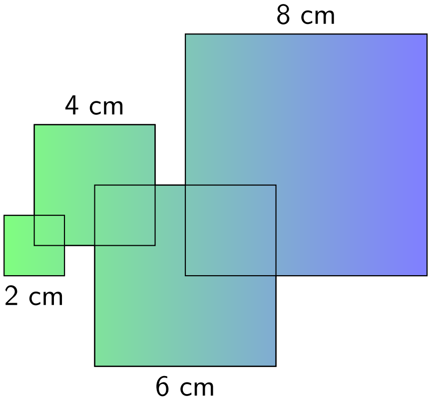
(a) 90 cm^2

(b) 92 cm^2

(c) 113 cm^2

(d) 120 cm^2

(e) None of the above



8. Let $x_1 = 1$ and $x_{n+1} = (-1)^n - x_n$ for each $n \geq 1$. Find x_{100}.

(a) -100

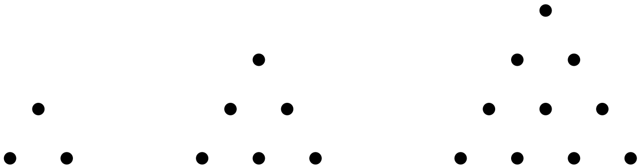
(b) -1

(c) 0

(d) 1

(e) 100

9. A triangular number is a number of dots that can be arranged in a triangular pattern such as the ones shown below. (So, 3, 6, and 10 are triangular numbers.) What is the smallest triangular number larger than 1 that is also a perfect square?



(a) 25

(b) 36

(c) 49

(d) 100

(e) 144

10. The solution set of the equation

$$(xy - 1)(xy)(y - x - 1) = 0$$

separates the plane into how many regions?

(a) 10

(b) 11

(c) 12

(d) 13

(e) 14

11. How many pairs of positive integers (a, b) satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{5} \ ?$$

(a) 0

(b) 1

(c) 2

(d) 3

(e) None of the above

12. Alex has two rectangular boxes. The length of the first box is 50% larger than that of the second box, the width of the first box is 20% larger than that of the second box, and the height of the first box is x% smaller than that of the second box. The volumes of the two boxes are the same. To the nearest integer, what is x?

(a) 66

(b) 55

(c) 44

(d) 33

(e) 22

13. If six regular dice (with 1 through 6 dots on their faces) are rolled, which of the following numbers is most likely to be the total number of dots on their top faces?

(a) 15

(b) 18

(c) 22

(d) 24

(e) 27

14. Evaluate:

$$\frac{4}{1} \cdot \frac{5}{2} \cdot \frac{6}{3} \cdot \cdots \cdot \frac{22}{19} \cdot \frac{23}{20}.$$

(a) 1,771

(b) 3,542

(c) 5,313

(d) 10,626

(e) None of the above

15. The line $x + y = 7$ intersects the circle $(x-2)^2 + (y-1)^2 = 10$ at points A and B. What is the length of the segment AB?

(a) 2

(b) $2\sqrt{2}$

(c) 3

(d) $3\sqrt{2}$

(e) 4

16. How many numbers in the arithmetic sequence

$$3, 7, 11, 15, \ldots, 2023$$

are perfect squares?

(a) 0

(b) 1

(c) 21

(d) 44

(e) None of the above

17. The degree measures of all three angles in a triangle are integers. What is the smallest possible value of the sum of degree measures of the smallest angle and the largest angle?

(a) 59

(b) 61

(c) 89

(d) 91

(e) 119

18. Given that $2023 = 7 \cdot 17^2$, find the sum of all positive factors of 2023.

(a) 24

(b) 314

(c) 2,047

(d) 2,337

(e) 2,456

19. Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

(a) 450

(b) 600

(c) 750

(d) 810

(e) 900

20. Alice is 40% older than Bob, and Bob is 20% younger than Cindy. The sum of their ages is 29.2 years. How old will Alice be on her next birthday?

(a) 12

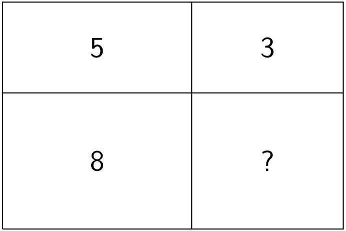
(b) 13

(c) 14

(d) 15

(e) None of the above

21. A rectangle is divided into four smaller rectangles as shown. The areas of three of these rectangles are 5 cm^2, 3 cm^2, and 8 cm^2. (Note: the picture below is not drawn to scale.) What is the area of the fourth rectangle?



(a) 4.5 cm^2

(b) 4.8 cm^2

(c) 5 cm^2

(d) Cannot be determined

(e) This is impossible

22. Let r and s be the roots of the equation $x^2 - 5x + 2 = 0$. Find $r^3 + s^3$.

(a) 95

(b) 78

(c) 42

(d) 39

(e) None of the above

23. What is the radius of a circle inscribed in a triangle with sides 7 in., 8 in., and 9 in.?

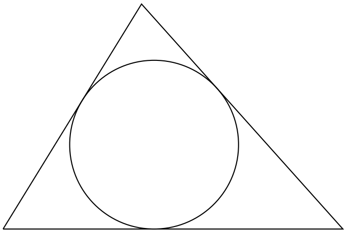
(a) 3 in.

(b) $2\sqrt{2}$ in.

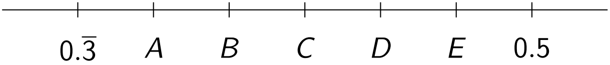
(c) $\sqrt{6}$ in.

(d) $\sqrt{5}$ in.

(e) None of the above



24. Points A, B, C, D, and E divide the interval $[0.\overline{3}, 0.5]$ into six segments of equal length. What number does point B represent?



(a) $0.3\overline{6}$

(b) $0.3\overline{8}$

(c) $0.3\overline{9}$

(d) $0.4\overline{1}$

(e) None of the above

25. The expression $\sqrt{20\sqrt{3}+37}$ can be simplified as which of the following?

(a) $\sqrt{3}+10$

(b) $5\sqrt{3}+1$

(c) $5\sqrt{3}+2$

(d) $2\sqrt{3}+5$

(e) $4\sqrt{3}+5$

26. When $(3x^4 - x)^{11}$ is fully expanded, what is the sum of all coefficients?

(a) -11

(b) 0

(c) 3

(d) 22

(e) 2048

27. In $\triangle ABC$, AD is the bisector of $\angle BAC$, $BD = 2$, and $DC = 4$. Which of the following cannot be the length of AD?

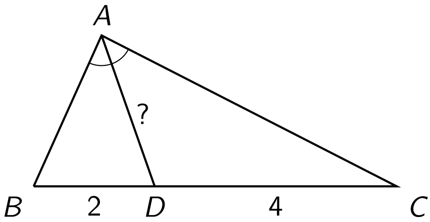
(a) 1

(b) 3

(c) 5

(d) 7

(e) 9



28. If $\cos(x) = 0$ and $\sin(x + y) = -\frac{1}{2}$, what is the smallest possible positive value of y?

(a) $\dfrac{\pi}{6}$

(b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{3}$

(d) $\dfrac{\pi}{2}$

(e) $\dfrac{2\pi}{3}$

29. The parabola $y = ax^2 + bx + c$ has vertex (d, d) and y-intercept $(0, -d)$, where $d \neq 0$. What is b?

(a) 4

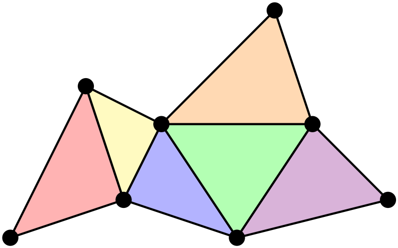
(b) 2

(c) 0

(d) -2

(e) -4

30. Little Lyn likes to draw pictures containing dots and line segments connecting some of these dots so that every finite region that is formed is a triangle. There are no dots that are not connected to any other dots. The picture shown below has 8 dots, 13 segments, and 6 triangles. Today she wants to draw a picture with 10 dots and 7 triangles. How many segments must she draw?



(a) 14

(b) 15

(c) 16

(d) 17

(e) This is impossible.

End of Part II

- Please remain seated until all Scantron forms have been collected.
- Meet your teacher, if you have arranged to do so.
- Lunch break until 1 pm.
- 1:00-2:30 pm: Games in Science II building, room 308.
- Approx. 2:45-3:45 pm: Awards ceremony in Science II courtyard.
    - Ties, if any, will be broken at the awards ceremony.

Answers

1. e
2. b
3. c
4. c
5. d
6. d
7. e
8. a
9. b
10. b
11. d
12. c
13. c
14. a
15. b

16. a
17. d
18. e
19. e
20. a
21. b
22. a
23. d
24. b
25. d
26. e
27. e
28. c
29. a
30. c