2023
Leap Frog Relay Grades 6-8

Solutions - Part I

1. Points P and R are located at $(2, 1)$ and $(12, 15)$, respectively. Point M is at the midpoint of segment $\overline{PR}$. Segment $\overline{PR}$ is reflected over the x-axis. What is the sum of the coordinates of the image of point M (the midpoint of the reflected segment)?

   (a) 2
   (b) -2
   (c) 0
   (d) 1
   (e) -1

   Solution. (e) Point M has coordinates $(7, 8)$. Therefore, its image has coordinates $(7, -8)$. Thus the sum is $7 - 8 = -1$.

2. The difference between the squares of two consecutive natural numbers is 27 more than the larger of the two natural numbers. What is the product of the two natural numbers?

   (a) 784
   (b) 730
   (c) 756
   (d) 792
   (e) 720

   Solution. (c) Let the smaller of the two natural numbers be n. Then the larger is $n + 1$. The difference between the squares of the two natural numbers is $(n + 1)^2 - n^2 = 2n + 1$. So $2n + 1 = (n + 1) + 27$. Solving the equation for n we get $n = 27$. Then the product of the two natural numbers is $27 \times 28 = 756$.

3. The numbers 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

(a) 12

(b) 18

(c) 24

(d) 36

(e) 48

Solution. **(b)** Each vertex appears on exactly 3 faces, so the sum of the numbers on all faces is $3(1+2+\cdots+8) = 108$. There are six faces for the cube, so the common sum must be $108/6 = 18$. A possible numbering is bottom clockwise $3, 6, 4, 5$, and top clockwise (starting from the vertex above 3 in the bottom) $8, 1, 7, 2$.

4. Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is $3@(6\#2)$?

(a) -304

(b) -296

(c) -48

(d) 36

(e) None of the above

Solution. **(a)** The quantity $6\#2 = 6 + 2 - 6(2)^2 = 8 - 24 = -16$. Therefore, $3@(6\#2) = 3@(-16) = 3(-16) - (-16)^2 = -48 - 256 = -304$.

5. Of the 500 balls in a large bag, 80% are green and the rest are brown. How many of the green balls must be removed from the bag so that 75% of the remaining balls are green?

(a) 25

(b) 125

(c) 75

(d) 100

(e) 50

Solution. **(d)** The number of brown balls in the bag is 20% of 500, which is 100. After x green balls are removed, the 100 brown balls must be 25%, or 1/4, of the balls in the bag, so there must be $4 \times 100 = 400$ balls in the bag. So $x = 500 - 400 = 100$ green balls must be removed.

6. Find the sum of all the distinct primes (without repetition) in the prime factorization of 2023.

(a) 24

(b) 36

(c) 42

(d) 30

(e) 48

Solution. **(a)** The prime factorization of 2023 is 7×17^2. Therefore the sum of the different primes is $7 + 17 = 24$.

7. Julia can ride her bicycle uphill at 5 miles per hour and downhill at 15 miles per hour. How far uphill in miles should she travel if she wants her round trip (uphill plus downhill) to last exactly half an hour?

(a) 1.525 miles

(b) 1.675 miles

(c) 1.875 miles

(d) 2.325 miles

(e) 2.375 miles

Solution. **(c)** If s is the traveled distance in miles, then the time spent going uphill is $s/5$ hours, and the time spent traveling downhill is $s/15$ hours. The sum of these two quantities should be $1/2$ hour. Solving the equation $s/5 + s/15 = 1/2$ gives $15/8 = 1.875$ (miles) for s.

8. Suppose that the four-digit number $74p8$ is divisible by 12. Find the sum of all possible values of the digit p.

(a) 15

(b) 10

(c) 7

(d) 20

(e) 12

Solution. **(b)** The number $74p8$ is divisible by 12 if it is divisible both by 3 and by 4. The number is divisible by 3 if the sum of its digits is divisible by 3. So $7 + 4 + p + 8 = 19 + p$ must be divisible by 3. This gives possible values of $2, 5$, or 8 for p. However, out of the 3 potential four-digit numbers only 7428 and 7488 are divisible by 4, too (the two-digit number formed by the last two digits must be divisible by 4). Therefore, the "good" values for p are 2 and 8, so their sum is $2 + 8 = 10$.

9. Jose has 9 blue socks, 18 black socks, and 10 white socks all mixed up in his drawer. It's late at night, and he doesn't want to turn on the light in the dark room. What is the smallest number of socks he has to pull out of his drawer to guarantee that he will have a pair of black socks?

(a) 17

(b) 18

(c) 19

(d) 20

(e) 21

Solution. (e) The worst-case scenario is that Jose pulls out all of the non-black socks first. There are $9 + 10 = 19$ such socks. However, the next 2 socks he pulls must be black since there are not any other colored socks left in the drawer. So among 21 socks there have to be at least 2 black socks. With a smaller number of socks it is not guaranteed.

10. Todd's friend Olivia is flying her plane at a constant elevation of 1.5 km. From the ground, Todd sees the plane moving in his direction from the west at a 30 degree angle of elevation. One minute later, after Olivia had flown directly overhead, Todd turns and sees the plane moving away from him to the east at a 45 degree angle of elevation. How fast is Olivia flying in kilometers per hour?

(a) About 230

(b) About 250

(c) About 270

(d) About 290

(e) About 310

Solution. (b) Let Todd stand at point A on the ground, and let point B represent the point in the sky that is exactly 1.5 km above Todd. Let C be the point where Todd spots Olivia's plane in the west, and let point D be the point where Todd sees Olivia's plane a minute later. From the given information we know that in triangle ABC the angle BAC is 60 degrees, and angle CBA is 90 degrees. Therefore, the triangle ABC is a "30-60-90" triangle. We also know that side AB is 1.5 km. Since in 30-60-90 triangles the shorter leg is half of the hypotenuse, AC must be 3 km. Applying the Pythagorean theorem, we can calculate that the length of side CB is the square root of $3^2 - (1.5)^2$, which is about 2.6 km. Since angle DAB is 45 degrees, the triangle ADB must be an isosceles right triangle, so $BD = AB = 1.5$ km. The approximate distance traveled by Olivia during one minute is $CD = CB + BD \approx 2.6 \text{ km } + 1.5 \text{ km } = 4.1 \text{ km}$. Therefore, Olivia's speed is about 4.1 km per minute, which is about 60×4.1 km per hour. That is about 246 km per hour, so it is closest to 250 kilometers per hour.

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Solutions - Part II

11. Sets A, B, and C all contain positive whole numbers that are less than 30 according to the definitions below:
Set $A = \{\text{multiples of } 4\}$
Set $B = \{\text{numbers that are 1 less than a prime}\}$
Set $C = \{\text{multiples of } 3\}$
Find the sum of the elements in the set $(A \cap C) \cup (B \cap C)$.

(a) 60
(b) 72
(c) 4
(d) 12
(e) 48

Solution. (a) Here are the following sets as defined above for the whole numbers $\{1, 2, \ldots, 29\}$:
$A = \{4, 8, 12, 16, 20, 24, 28\}$
$B = \{1, 2, 4, 6, 10, 12, 16, 18, 22, 28\}$
$C = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$
We have $A \cap C = \{12, 24\}$ and $B \cap C = \{6, 12, 18\}$. Thus, $(A \cap C) \cup (B \cap C) = \{6, 12, 18, 24\}$ and $6 + 12 + 18 + 24 = 60$.

12. When Demetri bought his car, he paid $1/4$ of the total price right away. To pay off the rest, each month he pays $1/5$ of the original price as a payment. How many monthly payments will it take Demetri to pay off his car? *Note that the final payment may be less than* $1/5$ *of the original price.*

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Solution. (d) Let x be the price of the car. Since he has paid off a quarter of the car, Demetri owes $\frac{3}{4}x$, so one can form the equation $n\left(\frac{1}{5}x\right) = \frac{3}{4}x$ to solve for the number of payments needed, n, to pay off the car. Since the x's cancel, one can solve and find: $n = \frac{5}{1} \cdot \frac{3}{4} = \frac{15}{4} = 3.75$, so it will take 4 payments to finish paying for the car.

13. Karla went to the store and bought three dozen eggs for \$4.80 per dozen. On her way home she met Tracy and gave Tracy back the \$6.00 she borrowed from her last week. She now has exactly half the money she had before going to the store. How much money does Karla have now?

(a) \$20.40

(b) \$14.40

(c) \$40.80

(d) \$18.00

(e) \$8.40

Solution. **(a)** Working in pennies and letting x be how much Karla originally has, then after going to the store for the eggs and also paying off Tracy, she has $x - 3 \cdot 480 - 600 = x - 2040$ left. So $x - 2040 = \frac{x}{2}$. Simplifying, we get $\frac{x}{2} = 2040$, which is what the problem was asking for since $\frac{x}{2}$ is half the money she had before going to the store.

14. When each side of a square was increased in length by 50%, its area increased by 180 square inches. How many square inches are in the original square?

(a) 240

(b) 144

(c) 100

(d) 90

(e) 80

Solution. **(b)** Given the original square has side length s, then the side length of the enlarged square is $s + 0.5s = 1.5s$. So the area of the enlarged square is $(1.5s)^2 = 2.25s^2$ which is equal to $s^2 + 180$. Solving gives $2.25s^2 - s^2 = 1.25s^2 = 180$. Thus the <u>area</u> of the original square is $s^2 = \frac{180}{1.25} = \frac{18000}{125} = 144$ square inches.

15. The ordered list of numbers $18, 21, 24, 26, A, 36, 37, B$ has a median of 30 and a mean of 30. Find $B - A$.

(a) 0

(b) 5

(c) 10

(d) 11

(e) None of the above

Solution. **(c)** Since there are eight numbers, the median is $\frac{26 + A}{2} = 30$ giving $A = 34$. Now finding the average by summing all the numbers and dividing by 8 gives $\frac{196 + B}{8} = 30 \implies 196 + B = 240 \implies B = 44$. Thus $B - A = 44 - 34 = 10$.

16. The sum of the digits of 2023 is $2+0+2+3=7$ and 7 is a factor of 2023. For how many numbers in the 2020's is the sum of the digits a factor of the number?

(a) 5

(b) 3

(c) 2

(d) 4

(e) 6

Solution. (e) Checking each of the possibilities of $2020, 2021, \ldots, 2029$ give the result of 6. For example for 2029, $2+0+2+9=13$ but 13 is not a factor of 2029 , while for 2025, $9 \cdot 225 = 2025$. The six solutions that satisfy the condition are $2020, 2022, 2023, 2024, 2025$, and 2028.

17. The product of the ages of three teenagers is 4590. How old is the oldest?

(a) 14

(b) 15

(c) 19

(d) 17

(e) 18

Solution. (e) The prime factorization of 4590 is $2^1 \times 3^3 \times 5^1 \times 17^1$. 17 itself is a "teenage number." The only way of breaking $2^1 \times 3^3 \times 5^1$ into two "teenage numbers" is 15 and 18.

18. Pat averages 12 MPH riding their bicycle to school. Averaging 36 MPH by car takes them one-half hour less time. How far do they travel to school?

(a) 15 miles

(b) 20 miles

(c) 12 miles

(d) 9 miles

(e) 36 miles

Solution. (d) Let d be the distance to school and t the number of hours it takes when riding a bike. Then $\dfrac{d}{t} = 12$, so $d = 12t$, and $\dfrac{d}{t - \frac{1}{2}} = 36$, so $d = 36\left(t - \dfrac{1}{2}\right) = 36t - 18$. Hence $36t - 18 = 12t$, so $24t = 18 \Longrightarrow t = \dfrac{18}{24} = \dfrac{3}{4}$ hours. Thus, $d = 12t = 12 \cdot \dfrac{3}{4} = 9$ miles.

19. The value of $3^3 = 27$. The *units digit* (ones place) for 3^3 is 7. What is the units digit for 3^{122}?

(a) 1

(b) 3

(c) 9

(d) 7

(e) 4

Solution. **(c)** Notice that in terms of units digits, $3^1 \to 3, 3^2 \to 9, 3^3 \to 7, 3^4 \to 1, 3^5 \to 3$ with the pattern continuing. So this eliminates the answer 4 as being a possible units digit. Since every 3^4 we get a units digit of 1 and $1 \cdot 1 \cdot \cdots \cdot 1$ is always 1, then we can take $\dfrac{122}{4} = 30R2$. So the units digit of $3^{122} = \left(3^4\right)^{30} \cdot 3^2$ is $1 \cdot 3^2 = 9$.

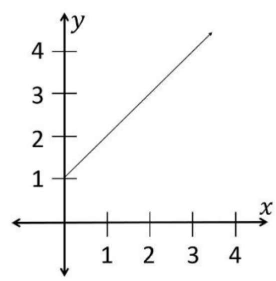
20. The following functions f, g, and h represent the distance, y, traveled at time x by three different objects (measured in the same units).

f:

x	y
0	1
5	6
10	11

g: $x = \frac{y}{4} - \frac{3}{8}$

h:



Which object is moving fastest?

(a) The object represented by function f.
(b) The object represented by function g.
(c) The object represented by function h.
(d) The objects represented by the functions f and h, which are traveling at the same speed.
(e) The answer cannot be determined with the information provided.

Solution. **(b)** We need the slopes of the three functions to determine the rates of change. Looking at the first function f, we know it is a line because the slopes are the same between any of the points. For example, $\dfrac{6-1}{5-0} = \dfrac{11-6}{10-5} = \dfrac{11-1}{10-0} = 1$. For function g, we solve for the y-intercept form by multiplying through by 8; we get $8x = 2y - 3 \Rightarrow 2y = 8x + 3 \Rightarrow y = 4x + \dfrac{3}{2}$, and now looking at the rise/run on the graph of the function h between two points, it appears that $(0, 1)$ and $(1, 2)$ are on the graph of the line, so the slope is approximately $\dfrac{2-1}{1-0} = 1$. So clearly the object moving in terms of function g is fastest with a rate of change of 4, versus 1 for the other two objects.