2015
Leap Frog Relay Grades 6-8
Part I Solutions

No calculators allowed
Correct Answer $= 4$ **points**
Incorrect Answer $= -1$ **point**
Blank = 0 points

1. The average age of 6 people in a room is 12 years. A 26-year-old person comes into the room. What is the average age of the 7 people in the room now?

   (a) 13 years
   (b) 14 years
   (c) 15 years
   (d) 19 years
   (e) It can't be calculated from the given information.

   Solution: (b)

   Originally the sum of the ages of the 6 people in the room is $6 \times 12 = 72$. After the 26-year-old person comes into the room, the sum of the ages of the people is $72 + 26 = 98$. So the average age of the seven people is $98/7 = 14$.

2. In the following equation, Z represents a missing exponent. Determine the value of Z:
$$\frac{9j^Z}{(-3j^2x^2)^5} = \frac{-1}{27j^7x^{10}}$$

(a) -1

(b) 0

(c) 1

(d) 2

(e) 3

Solution: (e)

$$\frac{9j^Z}{(-3j^2x^2)^5} = \frac{9j^Z}{(3^2)(-3^3)j^{10}x^{10}} = \frac{j^{Z-10}}{(-3^3))x^{10}} = \frac{j^{Z-10}}{-27x^{10}}.$$

Since $\frac{j^{Z-10}}{-27x^{10}} = \frac{-1}{27j^7x^{10}}$, it must be that $Z - 10 = -7$, so $Z = 3$.

3. Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two cards of the same color or two cards of the same letter. What is the probability of drawing a winning pair?

(a) $\frac{2}{7}$

(b) $\frac{3}{8}$

(c) $\frac{1}{2}$

(d) $\frac{4}{7}$

(e) $\frac{5}{8}$

Solution: (d)

After the first card is dealt, there are seven left. The three cards with the same color as the initial card are winners and so is the card with the same letter but different color. That means four of the remaining seven cards form winning pairs with the first card, so the probability of winning is $\frac{4}{7}$.

4. A car's gasoline tank is 3/5 full. After adding 3 liters of gas, the gauge shows that the tank is 3/4 full. How many liters of gas does the tank hold when full?

(a) 15 liters
(b) 18 liters
(c) 20 liters
(d) 24 liters
(e) 21 liters

Solution: (c)
Let x be the amount of gas the tank holds. Then $\frac{3}{5}x + 3 = \frac{3}{4}x$
$\frac{3}{4}x - \frac{3}{5}x = 3$

$\frac{15}{20}x - \frac{12}{20}x = 3$

$\frac{3}{20}x = 3$

$x = \frac{20}{3} \cdot 3 = 20$ liters.

5. I gave a value to every vertex of a cube. The value of an edge is the sum of the values of the vertices at its ends. The value of a side is the sum of the values of the edges surrounding it. The value of a cube is the sum of the values of its sides. What is the value of the cube if the sum of the values of its vertices is 18?

(a) 18
(b) 36
(c) 108
(d) 516
(e) 768

Solution: (c)

There are 3 edges starting from each vertex of the cube, therefore the sum of the values of the edges is 3 times the sum of the values of the vertices. Each edge is counted as a border of 2 sides, therefore the sum of the values of the sides is 2 times the sum of the values of the edges, so 6 times the sum of the values of the vertices: $6 \times 18 = 108$.

6. Three horses race. In how many ways can they finish if ties are possible?

   (a) 6
   (b) 8
   (c) 9
   (d) 10
   (e) 13

   Solution: (e)

   If there are no ties, then there are $3! = 6$ possible ways to finish. If there are ties between two horses, there are 2 different outcomes (the two tied horses can be tied for the 1st place or the 2nd place) and for each of these there are 3 ways to finish. If they all tie, there is only 1 way to finish. So the total number of ways to finish is $6 + 6 + 1 = 13$.

7. Suppose that x, y, and z are positive numbers, and $(x+y+z)^x = 4$, $(x+y+z)^y = 2$, and $(x+y+z)^z = 32$. Find z.

   (a) $\frac{2}{3}$
   (b) $\frac{5}{3}$
   (c) $\frac{7}{4}$
   (d) $\frac{3}{2}$
   (e) $\frac{5}{2}$

   Solution: (e)

   $(x+y+z)^{x+y+z} = (x+y+z)^x \cdot (x+y+z)^y \cdot (x+y+z)^z = 4 \cdot 2 \cdot 32 = 4 \cdot 4 \cdot 4 \cdot 4 = 4^4$

   $x + y + z = 4$

   $4^z = 32$

   $2^{2z} = 2^5$

   $2z = 5$

   $z = \frac{5}{2}$.

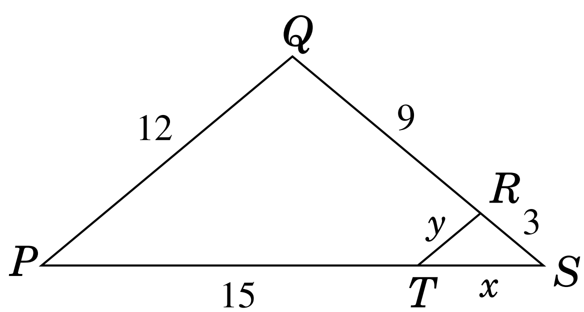
8. At today's exchange rate, two British pounds (£2) is equal to \$3.02. Rounding to one decimal place, how many British pounds would be needed to buy a bicyle costing \$210?

   (a) £143.5
   (b) £139.1
   (c) £317.1
   (d) £317.09
   (e) £143.46

   Solution: (b)

   To convert from dollars to pounds, use the conversion factor $\frac{\pounds 2}{\$3.02}$ as follows: $\$210 \cdot \frac{\pounds 2}{\$3.02} = \frac{\pounds 420}{3.02} = \pounds 139.0728... \approx \pounds 139.1$ after some long division and rounding.

9. In the picture below, $\triangle PQS \sim \triangle TRS$. Determine $(y - x)^3$.



   (a) −8
   (b) −27
   (c) 27
   (d) −64
   (e) 125

   Solution: (a)

Since the triangles are similar, $\frac{x}{x+15} = \frac{3}{3+9} = \frac{1}{4}$

$4x = x + 15$

$3x = 15$

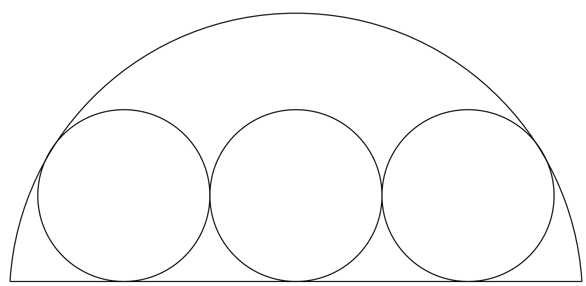
$x = 5.$

Also, $\frac{y}{12} = \frac{3}{12}$

$y = 3$

Then $(y - x)^3 = (3 - 5)^3 = (-2)^3 = -8.$

10. In the picture below, three small circles, each of radius 1, lie inside a larger semicircle. Each small circle is tangent to the diameter of the large semicircle. The two outer small circles are each tangent to the semicircle, and the inner small circle is tangent to the other two. What is the radius of the large semicircle?



(a) $1 + \sqrt{5}$

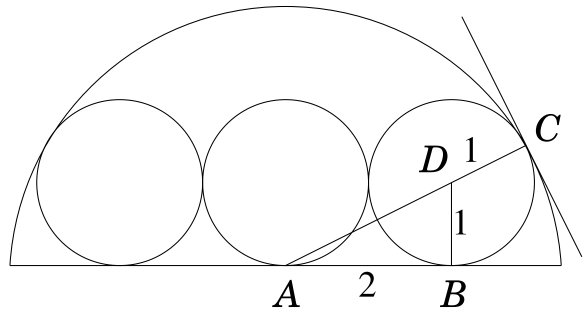
(b) $\pi + 1$

(c) 3

(d) $\sqrt{7}$

(e) $\sqrt{3} + 2$

Solution: (a)



Let A and D be centers of circles, and B and C points of tangency, as shown above. Then the line perpendicular to the circles at the point of tangency C passes through D and A. By the Pythagorean Theorem, AD has length $\sqrt{5}$ and DC has length 1, so radius AC has length $1+\sqrt{5}$.

2015
Leap Frog Relay Grades 6-8
Part II Solutions

No calculators allowed
Correct Answer $= 4$ **points**
Incorrect Answer $= -1$ **point**
Blank = 0 points

11. For any positive integer n, define S(n) to be the sum of the positive factors of n. For example, $S(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Find $S(S(10))$.

    (a) 8
    (b) 9
    (c) 18
    (d) 28
    (e) 39

    Solution: (e)

    The positive factors of 10 are 1, 2, 5, 10, therefore $S(10) = 1 + 2 + 5 + 10 = 18$. Therefore, $S(S(10)) = S(18)$. The sum of the factors of 18 is $1 + 2 + 3 + 6 + 9 + 18 = 39$, so $S(S(10)) = S(18) = 39$.

12. Carla has 60 ribbons, of which 25% is red, 30% is white, and 45% is blue. If she buys 15 more white ribbons, 12 more blue ribbons, and 3 more red ribbons, what percent of her ribbons will be red?

    (a) 5

(b) 15

(c) 20

(d) 25

(e) 30

Solution: (c)

Carla had $0.25 \times 60 = 15$ red ribbons. If she buys 30 more ribbons, (three of them were red), she will have $60 + 30 = 90$ ribbons. The 18 red ribbons (15 old and 3 new) will be $18/90 \times 100 = 0.2 \times 100 = 20$ percent of the total number of ribbons.

13. Let a, b, and c be numbers with $0 < a < b < c$. Which of the following is impossible?

    (a) $a + c < b$

    (b) $ab < c$

    (c) $a + b < c$

    (d) $ac < b$

    (e) $b/c = a$

    Solution: (a)

    Since c is the largest among the three positive numbers, $a + c$ must be larger than b. The other choices are all possible. For example, (b) and (c) are possible if $a = 1,\ b = 2$, and $c = 4$. Choice (d) is possible if $a = 1/3,\ b = 1,\ c = 2$. Choice (e) is possible if $a = \frac{1}{2},\ b = 1,\ c = 2$.

14. A haunted house has five windows. In how many ways can Georgie the Ghost enter the house by one window and leave by a different window?

    (a) 5

    (b) 9

    (c) 15

    (d) 20

(e) 120

Solution: (d)

Georgie has five choices for the window through which to enter. After entering, Georgie has four choices for the window through which to exit. So altogether there are $5 \times 4 = 20$ different ways for Georgie to enter by one window and exit by another.

15. The base of an isosceles triangle is 24 units and its area is 60 square units. What is the length of one of the congruent sides?

    (a) 5 units
    (b) 12 units
    (c) 13 units
    (d) 36 units
    (e) 84 units

    Solution: (c)

    The area of the triangle can be calculated as base times height divided by 2. Since the base is 24 units, and the area is 60 square units, it means the height must be 5 units. The height cuts the original triangle into two congruent parts, both of which are right triangles with legs 5 and 12 units and the hypotenuses being the two congruent sides of the original isosceles triangle (draw a picture). Using the Pythagorean theorem, $5^2 + 12^2 = 25 + 144 = 169 = 13^2$, so the length of the hypotenuse is 13 units.

16. Express the number $2^{-3} \cdot 3^2 \cdot 4^3 \cdot 5^2$ in scientific notation.

    (a) 143200
    (b) 1.432×10^5
    (c) 1.4×10^{10}
    (d) 1.8×10^3
    (e) 1.2×10^3

Solution: (d)

$2^{-3} \cdot 3^2 \cdot 4^3 \cdot 5^2 = \frac{14400}{8} = 1800 = 1.8 \times 10^3$.

17. In an airplane contest, a jet plane traveling at 500 mph overtakes a propeller plane traveling at 200 mph that had a 3-hour head start. How far from the starting point are the two planes when they meet?

    (a) 375 miles
    (b) 3020 miles
    (c) 285 miles
    (d) 1000 miles
    (e) 600 miles

    Solution: (d)

    Since $\dfrac{\text{distance}}{\text{time}} = \text{rate}$, distance = rate $\cdot$ time, so
    $500 \cdot t = 200 \cdot (t + 3)$
    $500t = 200t + 600$
    $300t = 600$
    $t = 2$
    The distance is $500 \cdot 2 = 1000$ miles.

18. Given the formula $T = \sqrt{s(s-a)(s-b)(s-c)}$, solve for b.

    (a) $\frac{T}{s(s-a)(s-c)} + s$
    (b) $s - \frac{T^2}{s(s-a)(s-c)}$
    (c) $T^2 + \sqrt{s+c}$
    (d) $T^2 - \sqrt{s\,(s+c)}$
    (e) $s + \frac{T^2}{s(s-a)+c}$

    Solution: (b)

First square both sides to get $T^2 = s(s-a)(s-b)(s-c)$.
Then divide as follows, $\frac{T^2}{s(s-a)(s-c)} = s - b$.
Now we may multiply both sides by -1 to get $-\frac{T^2}{s(s-a)(s-c)} = b - s$.
The final step of adding s to both sides gives $b = s - \frac{T^2}{s(s-a)(s-c)}$.

19. A cylinder-shaped coffee can has a 5 cm radius and an 8 cm height. Calculate the surface area of the can.

    (a) 40π cm^2
    (b) 80π cm^2
    (c) 110π cm^2
    (d) 120π cm^2
    (e) 130π cm^2

    Solution: (e)

    The area of the top of the can is 25π cm^2, the area of the bottom is also 25π cm^2. The area of surface connecting the top with the bottom is $2\pi \cdot 5 \cdot 8$ cm$^2 = 80\pi$ cm^2. The surface area of the can is the sum of these three quantities: 130π cm^2.

20. Two bicycle clubs organize a tour together. At the meeting in the morning all members great each other with a handshake. Everybody shakes hands with everybody else once. There were a total of 231 handshakes of which 119 happened between members of the same club. The bigger club has how many more members than the smaller one?

    (a) 6
    (b) 7
    (c) 8
    (d) 9
    (e) 10

    Solution: (a)

Let n be the total number of participants from the two clubs. Since every person shook hands with everybody else, $\frac{n(n-1)}{2} = 231$, so $n = 22$. If there are x members in one club, then there are $22 - x$ members in the other club. The number of handshakes among members of each club are $\frac{x(x-1)}{2}$ and $\frac{(22-x)(22-x-1)}{2}$.

Then $\frac{x(x-1)}{2} + \frac{(22-x)(22-x-1)}{2} = 119$. Solving for x gives $x = 8$ or $x = 14$. Therefore, there are 8 members in one club, and 14 members in the other club. The bigger club has $14 - 8 = 6$ more members.