Mad Hatter 11-12. Part I.

Math Field Day. California State University, Fresno.
April 21st, 2012.

Question 1.

A square has perimeter P and area A. If $A = 3P$, then what is the value of P?

(a) 24
(b) 32
(c) 36
(d) 48
(e) 54

Question 2.

A ladder was resting against a wall. A kitten sits at the middle step, and the ladder slides down the wall. What is the trajectory of the kitten?

(a) A straight line segment
(b) An arc of a circle
(c) A piece of a parabola
(d) A piece of a hyperbola
(e) None of the above

Question 3.

The average of 10 numbers is 100. One of the numbers in the list is deleted, and the resulting average of nine numbers is equal to 101. What is the value of the deleted number?

(a) 10
(b) 90
(c) 91
(d) 99
(e) 100

Question 4.

At a party, every two people shook hands once. How many people attended the party if there were exactly 210 handshakes?

(a) 209
(b) 105
(c) 41
(d) 20
(e) None of the above

Question 5.

In hexadecimal number system: $49_{16} + \_\_\_\_{16} = 125_{16}$.

(a) 76
(b) 7C
(c) A6
(d) AC
(e) None of the above

Question 6.

The angles of a hexagon have degree measures that are six consecutive odd numbers. What is the degree measure of the smallest angle?

(a) 55
(b) 59
(c) 115
(d) 119
(e) None of the above

Question 7.

How many real zeros does the polynomial $p(x) = -6x^4 + 11x^3 - 6x^2 + x$ have?

(a) 0
(b) 1
(c) 2
(d) 4
(e) Not enough information to tell.

Question 8.

If you roll three dice, what is the probability of the sum being at least 11?

(a) 50%
(b) 62%
(c) 74%
(d) 75%
(e) None of the above

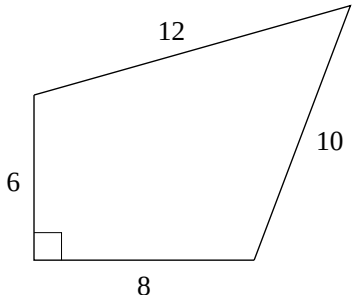
Question 9.

For how many integers m does there exist at least one integer n such that $mn \leq m + n$?

(a) 3
(b) 4
(c) 6
(d) 12
(e) Infinitely many

Question 10.

Find the area of the quadrilateral shown below.



(a) 48
(b) 64
(c) 72
(d) 80
(e) None of the above

Question 11.

What is the value of $\sqrt{10 + 4\sqrt{6}} - \sqrt{10 - 4\sqrt{6}}$?

(a) 1
(b) 4
(c) $2\sqrt{6}$
(d) $\sqrt{8\sqrt{6}}$
(e) $8\sqrt{6}$

Question 12.

How many two digit prime numbers are there in which both digits are prime numbers?

(a) 3
(b) 4
(c) 5
(d) 8
(e) 14

Question 13.

What is the value of $(\log_2 3^2)(\log_3 4^3)(\log_4 5^4)\cdots(\log_{63} 64^{63})$?

(a) $\frac{63!}{6}$
(b) $63!$
(c) 6
(d) $6 \cdot 63!$
(e) 32

Question 14.

On a test the passing students had an average of 83, while the failing students had an average of 55. If the overall class average was 76, what percent of the class passed?

(a) 44%
(b) 66%
(c) 68%
(d) 72%
(e) 75%

Question 15.

Oscar and Carmen walk around a circular track. It takes Oscar and Carmen respectively 6 and 10 minutes to finish each lap. They start at the same time, at the same point on the track, and walk in the same direction around the track. After how many minutes will they be at the same spot again (not necessarily at the starting point) for the first time after they start walking?

(a) 15
(b) 16
(c) 30
(d) 32
(e) 60

Question 16.

Determine the real part of $(1+i)^5$.

(a) 1
(b) $4\sqrt{2}$
(c) 21
(d) 32
(e) None of the above

Question 17.

When the length of a rectangle is increased by 10% and the width is decreased by 10%, then the area is

(a) Unchanged
(b) Increased by 10%
(c) Decreased by 10%
(d) Increased by 1%
(e) Decreased by 1%

Question 18.

What is the units digit of 2^{2012}?

(a) 0
(b) 2
(c) 4
(d) 6
(e) 8

Question 19.

If an arc of 60° on circle A has the same length as an arc of 30° on circle B, then the ratio of the area of circle A to the area of circle B is

(a) 1 : 6
(b) 1 : 4
(c) 1 : 3
(d) 1 : 2
(e) None of the above

Question 20.

How many even positive divisors does $2^5 \cdot 3^5$ have?

(a) 10
(b) 20
(c) 30
(d) 36
(e) None of the above

Question 21.

If $\sin(x) + \cos(x) = 1/2$, what is the value of $\sin^3(x) + \cos^3(x)$?

(a) $\frac{1}{8}$
(b) $\frac{5}{16}$
(c) $\frac{3}{8}$
(d) $\frac{5}{8}$
(e) $\frac{11}{16}$

Question 22.

Which of the following is the largest number?

(a) $2^{(3^4)}$
(b) $4^{(3^2)}$
(c) $8^{(4^2)}$
(d) $(16^8)^2$
(e) $(32^2)^8$

Question 23.

The two roots of the quadratic equation $x^2 - 85x + c = 0$ are prime numbers. What is the value of c?

(a) 84
(b) 166
(c) 332
(d) 664
(e) 1328

Question 24.

Find the area of the region in $\mathbb{R}^2$ consisting of all the points that satisfy

$$\begin{cases} x^2+y^2 & \geq & 1 \\ (x-1)^2+(y-1)^2 & \leq & 9 \end{cases}$$

(a) π
(b) 2π
(c) 8π
(d) 80π
(e) None of the above

Question 25.

How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Question 26.

What is the value of $\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{4\pi}{5}\right)+\sin\left(\frac{6\pi}{5}\right)+\sin\left(\frac{8\pi}{5}\right)$?

(a) 1
(b) -1
(c) 0
(d) $\frac{1}{\sqrt{5}}$
(e) $-\frac{1}{\sqrt{5}}$

Question 27.

$111_5 = \_\_\_\_\_{}_6$.

(a) 11
(b) 15
(c) 51
(d) 55
(e) None of the above

Question 28.

How many different prime factors does the number 20! have?

(a) 6
(b) 8
(c) 19
(d) 20
(e) None of the above

Question 29.

The set of all points (x, y) in $\mathbb{R}^2$ that satisfy $(x + y + 1)(x + y + 2) = 0$ consists of

(a) a single dot
(b) two parallel lines
(c) two intersection lines
(d) a circle
(e) None of the above

Question 30.

Evaluate: $\dfrac{2013^2 - 2011^2}{2010 \cdot 2015 - 2009 \cdot 2014}$

(a) $\frac{1}{8}$
(b) $\frac{1}{4}$
(c) 1
(d) 2
(e) None of the above

Question 31.

What is the largest integer n such that $\dfrac{n^2 - 2012}{n + 1}$ is an integer?

(a) 2010
(b) 2011
(c) 2012
(d) 1001
(e) None of the above.

Question 32.

If three distinct integer numbers have a sum of 10 and a product of 20, what is the median of the three numbers?

(a) 3
(b) 4
(c) 5
(d) 6
(e) There is not enough information given.

Question 33.

For a positive integer n, define $S(n)$ to be the sum of the positive divisors of n. Which of the following is the smallest?

(a) $S(2010)$
(b) $S(2011)$
(c) $S(2012)$
(d) $S(2013)$
(e) $S(2014)$

Question 34.

How many roots does the equation $\sqrt{x^2+6}+\sqrt{x^2+5}=4$ have?

(a) 0
(b) 1
(c) 2
(d) 3
(e) None of the above

Question 35.

Compute: $1+2+3+\ldots+2012$.

(a) $2,018$
(b) $2,025,078$
(c) $3,152,265$
(d) $4,050,156$
(e) None of the above

Question 36.

Convert $1,234_{25}$ to base 5.

(a) $2,468$
(b) $12,340$
(c) $1,020,304$
(d) $2,040,608$
(e) None of the above

Question 37.

Given that $4^x + 4^{-x} = 7$, find $2^x + 2^{-x}$.

(a) 2
(b) 3
(c) 4
(d) 5
(e) 6

Question 38.

Today is Saturday. What day of the week will be exactly 2012 days from today?

(a) Saturday
(b) Sunday
(c) Monday
(d) Tuesday
(e) Wednesday

Question 39.

How many intersection points do the line $3x + 2y = 10$ and the circle $(x+1)^2 + (y+1)^2 = 1$ have?

(a) 1
(b) 2
(c) 3
(d) Infinitely many
(e) None of the above

Question 40.

You own seventeen pairs of socks, all different, and all of the socks are individually jumbled in a drawer. One morning you rummage through the drawer and continue to pull out socks until you have a matching pair. How many socks must you pull out to guarantee having a matching pair?

(a) 3
(b) 8
(c) 15
(d) 18
(e) 25

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Question 41.

Find the natural number n such that

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \ldots \cdot \log_n(n+1) = 5.$$

(a) 3
(b) 7
(c) 15
(d) 31
(e) Does not exist

Question 42.

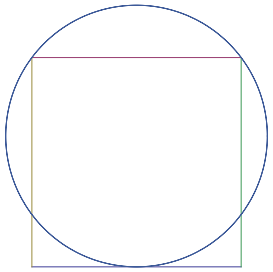
Which of the following figures has the largest area?

(a) A right triangle with legs 4 and 16
(b) A square with side 6
(c) A circle with radius 3
(d) A rectangle with sides 3 and 11
(e) A trapezoid with sides 5, 5, 5, and 11.

Question 43.

A circle passes through two adjacent vertices of a square and is tangent to one side of the square. If the side length of the square is 2, what is the radius of the circle?

(a) $\frac{3}{2}$
(b) $\frac{4}{3}$
(c) $\frac{5}{4}$
(d) $\frac{6}{5}$
(e) None of these.



Question 44.

If x and y are positive real numbers, neither of which is equal to 1, what is the smallest nonnegative value of $\log_x(y) + \log_y(x)$?

(a) 0
(b) $\sqrt{2}$
(c) $\sqrt{\pi}$
(d) 2
(e) 10

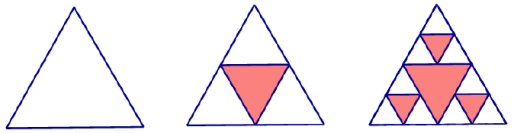
Question 45.

A class has three girls and three boys. These students line up at random, one after another. What is the probability that no boy is right next to another boy, and no girl is right next to another girl?

(a) $\frac{1}{20}$
(b) $\frac{1}{12}$
(c) $\frac{1}{10}$
(d) $\frac{3}{10}$
(e) $\frac{1}{2}$

Question 46.

The first figure in the sequence below is an equilateral triangle. The second has an inverted (shaded) equilateral triangle inscribed inside as shown. Each subsequent figure in this sequence is obtained by inserting an inverted (shaded) triangle inside each non-inverted (white) triangle of the previous figure, as shown below. How many regions (both shaded and white together) are in the ninth figure in this sequence?



(a) 4022
(b) 4023
(c) 9841
(d) 9842
(e) 4021

Question 47.

How many positive integers n have the property that when $1,000,063$ is divided by n, the remainder is 63?

(a) 29
(b) 37
(c) 39
(d) 49
(e) 79

Question 48.

All of the positive integers are written in a triangular pattern, beginning with the following four lines and continuing in the same way:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & 3 & 4 & & \\ & 5 & 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array}$$

Which number appears directly below 2012? (*Hint:* Each number in the $2n^{th}$ row is equal to $2n-2$ plus the number above it)

(a) 2100
(b) 2102
(c) 2104
(d) 2106
(e) 2108

Question 49.

Two spies agreed to meet at a gas station between noon and 1pm, but they have both forgotten the arranged time. Each arrives at a random time between noon and 1pm and stays for 6 minutes unless the other is there before the 6 minutes are up. Assuming all random times are equally likely, what is the probability that they will meet within the hour (noon to 1pm)?

(a) 0.12
(b) 0.15
(c) 0.17
(d) 0.19
(e) 0.25

Question 50.

How many triples (x, y, z) of rational numbers satisfy the following system of equations?

$$\begin{aligned} x+y+z &= 0 \\ xyz+z &= 0 \\ xy+yz+xz+y &= 0 \end{aligned}$$

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Question 51.

Suppose $f(x) = ax + b$ where a and b are real numbers. We define

$$f_1(x) = f(x)$$

and

$$f_{n+1}(x) = f(f_n(x))$$

for all positive integers n. If $f_7(x) = 128x + 381$, what is the value of $a + b$?

(a) 1
(b) 2
(c) 5
(d) 7
(e) 8

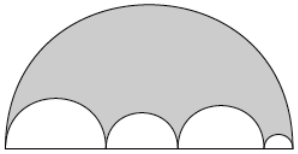
Question 52.

Find $\tan\left(\arcsin\left(-\frac{4}{5}\right) - \arccos\left(-\frac{5}{13}\right)\right)$.

(a) $\frac{25}{63}$
(b) $-\frac{3}{7}$
(c) $\frac{4}{13}$
(d) $-\frac{33}{56}$
(e) $\frac{16}{63}$

Question 53.

In the figure shown, five semicircles are all centered on the line and the sum of the diameters of the four smaller semicircles is 2 (which is the diameter of the big semicircle). What is the area of the square whose perimeter equals the perimeter of the shaded region?



(a) 2
(b) 4
(c) 2π
(d) π^2
(e) $\frac{\pi^2}{4}$

Question 54.

Find the number of integer pairs (x, y) satisfying the inequality $1 < |x| + |y| \leq 2$

(a) 4
(b) 5
(c) 8
(d) 12
(e) 13

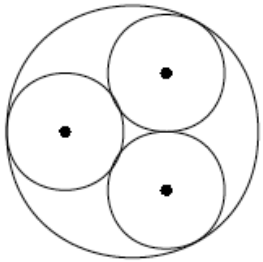
Question 55.

The equation $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$ has four roots. Denote the sum of the roots by s and the product of the roots by p. Find $\frac{s}{p}$.

(a) $-4/5$
(b) $3/5$
(c) $-2/5$
(d) 1
(e) $-1/5$

Question 56.

In the figure shown, three small circles with radius r are tangent to each other. The big circle with radius R is tangent to all three small circles.



What is the value of r/R?

(a) $2\sqrt{3} - 3$
(b) $1/2$
(c) $\frac{\sqrt{3}}{2}$
(d) $\sqrt{3} - 1$
(e) $2 - \sqrt{3}$

Question 57.

Find the sum

$$\sum_{k=100}^{120} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}$$

(a) $9/10$
(b) $10/11$
(c) $1/55$
(d) $1/100$
(e) None of the above

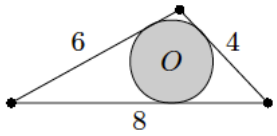
Question 58.

Given that $0.475 < \log_{10} 3 < 0.478$, how many digits are in the number 3^{50}?

(a) 50
(b) 36
(c) 24
(d) 23
(e) 18

Question 59.

In the figure shown, circle O is inscribed in the triangle with sides 4, 6 and 8. Find the area of the inscribed circle.



(a) $\frac{5\pi}{3}$
(b) $\frac{4\pi}{3}$
(c) 2π
(d) π
(e) None of these.

Question 60.

Denote by A the sum of the infinite series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.
In terms of A, what is the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$?

(a) $A/2$
(b) $A/3$
(c) $A/4$
(d) $3A/4$
(e) $2A/3$

Question 61.

If $a^2 - 3a + 1 = 0$, find $\frac{a^3}{a^6 + 1}$.

(a) 1/15
(b) 1/16
(c) 1/17
(d) 1/18
(e) 1/19

Question 62.

If the distance from the point (m^2, m) in Quadrant I to the line $y = x - 2$ is $2\sqrt{2}$, what is the value of m?

(a) 3
(b) 2
(c) 1
(d) 4
(e) 5

Question 63.

Suppose x and y are real solutions to the equation $\sqrt{x+1}+\sqrt{y-x}=0$. What is the value of $|x+2y|$?

(a) 1
(b) 2
(c) 3
(d) 4
(e) None of the above.

Question 64.

If $\sqrt{1+\sqrt{2+\sqrt{x}}}=3$, find the sum of the digits of x.

(a) 4
(b) 8
(c) 13
(d) 19
(e) None of the above

Question 65.

Convert 2012_3 to base 10.

(a) 23
(b) 32
(c) 59
(d) 95
(e) None of the above

Question 66.

The function

$$f(x)=1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$$

is well defined for all x except $x=0$, $x=-1$ and

(a) 1
(b) -2
(c) -3
(d) $-\frac{1}{2}$
(e) $-\frac{1}{3}$

Question 67.

What is the last digit of 2^{2012}?

(a) 0
(b) 6
(c) 4
(d) 2
(e) 8

Question 68.

If $\frac{x+4}{3} = \frac{y+3}{2} = \frac{z+8}{4}$ and $x+y+z = 12$, what is the value of $(x-y)^z$?

(a) 1
(b) 4
(c) 9
(d) 16
(e) 32

Question 69.

What is the remainder when the polynomial $2012x^{2012} + 2010x^{2010} + 1$ is divided by $x^2 + 1$?

(a) -1
(b) $x - 1$
(c) 1
(d) 2
(e) None of these

Question 70.

The equation $3^{x^2} = 81^{2x-3}$ has two solutions. What is their sum?

(a) -2
(b) 2
(c) 3
(d) 6
(e) 8

Question 71.

Let

$$f(x,y) = \binom{10}{1}x^9y - \binom{10}{3}x^7y^3 + \binom{10}{5}x^5y^5 - \binom{10}{7}x^3y^7 + \binom{10}{9}xy^9.$$

How many pairs of real numbers (x, y) are there such that $x^2 + y^2 = 1$ and $f(x, y) = 0$?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

Question 72.

Suppose that x and y are positive numbers. If $\frac{x+y}{x} = \frac{x}{y}$, then $\frac{2y}{x}$ is

(a) $\sqrt{5} - 1$
(b) $4/3$
(c) $3 - \sqrt{5}$
(d) $6/5$
(e) None of these.

Question 73.

Find the radius of the circle inscribed in a triangle with sides 3, 4, 5.

(a) $\frac{3}{4}$
(b) 1
(c) $\frac{5}{4}$
(d) $\frac{5}{3}$
(e) None of the above

Question 74.

The diagonals of a trapezoid divide it into four triangles. Find the area of the trapezoid if the areas of the two triangles adjacent to the bases of the trapezoid are S_1 and S_2.

(a) $S_1 + S_2$
(b) $S_1 \cdot S_2$
(c) $S_1 + S_2 + S_1 \cdot S_2$
(d) $\sqrt{S_1} \cdot \sqrt{S_2}$
(e) $(\sqrt{S_1} + \sqrt{S_2})^2$

Question 75.

Two parallel chords in a circle have lengths 8 and 10. The distance between them is 2. Then the radius of the circle is

(a) 5
(b) $1.5\sqrt{15}$
(c) 7.5
(d) $1.25\sqrt{17}$
(e) None of the above

Question 76.

If a, b, $a+b$, and $a-b$ are all prime numbers, which of the following statements must be true about the sum of these four numbers?

(a) The sum is odd and prime.
(b) The sum is odd and divisible by 3.
(c) The sum is odd and divisible by 7.
(d) The sum is even.
(e) None of the above.

Question 77.

An urn contains three white, four black, and five red balls. We take out a ball and put it in a drawer without looking at it. After that we take out a second ball. Find the probability that the second ball is white.

(a) $\frac{1}{4}$
(b) $\frac{1}{6}$
(c) $\frac{1}{3}$
(d) $\frac{3}{11}$
(e) $\frac{2}{11}$

Question 78.

If i is the imaginary number, what is i^{2012}?

(a) 1
(b) -1
(c) i
(d) $-i$
(e) None of the above

Question 79.

A pair of dice is thrown. What is the probability that the two numbers that appear differ by exactly 3?

(a) $\frac{2}{3}$
(b) $\frac{1}{3}$
(c) $\frac{1}{6}$
(d) $\frac{2}{9}$
(e) $\frac{1}{9}$

Question 80.

What is the volume of the cube that circumscribes the sphere that circumscribes the cube that circumscribes the sphere of radius 1 inch?

(a) $9\sqrt{3}$ in^3
(b) $16\sqrt{2}$ in^3
(c) $24\sqrt{3}$ in^3
(d) $54\sqrt{2}$ in^3
(e) None of the above

Answer Key: Part I.

(1) *d*	(2) *b*	(3) *c*	(4) *d*	(5) *e*	(6) *c*	(7) *d*	(8) *a*
(9) *e*	(10) *c*	(11) *b*	(12) *b*	(13) *d*	(14) *e*	(15) *b*	(16) *e*
(17) *e*	(18) *d*	(19) *b*	(20) *c*	(21) *e*	(22) *a*	(23) *b*	(24) *c*
(25) *c*	(26) *c*	(27) *c*	(28) *b*	(29) *b*	(30) *d*	(31) *a*	(32) *b*
(33) *b*	(34) *a*	(35) *b*	(36) *c*	(37) *b*	(38) *d*	(39) *e*	(40) *d*

Answer Key: Part II.

(41) *d*	(42) *b*	(43) *c*	(44) *d*	(45) *c*	(46) *c*	(47) *b*	(48) *b*
(49) *d*	(50) *b*	(51) *c*	(52) *e*	(53) *e*	(54) *c*	(55) *c*	(56) *a*
(57) *e*	(58) *c*	(59) *a*	(60) *d*	(61) *d*	(62) *a*	(63) *c*	(64) *d*
(65) *c*	(66) *d*	(67) *b*	(68) *d*	(69) *e*	(70) *e*	(71) *d*	(72) *a*
(73) *b*	(74) *e*	(75) *d*	(76) *a*	(77) *a*	(78) *a*	(79) *c*	(80) *c*