2010
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. Let g be the greatest common divisor of 1776 and 2010. Then...

   (a) $1 \leq g \leq 5$ (b) $6 \leq g \leq 10$

   (c) $11 \leq g \leq 15$ (d) $16 \leq g$

   (e) None of these

   Solution. (b) Factor $1776 = 2^4 \cdot 3 \cdot 37$ and $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Then

$$\begin{aligned} \gcd(1776, 2010) &= 2 \cdot 3 \\ &= 6. \end{aligned}$$

2. Happy Feet sells sandals 20% below list price. The bargain bin of sandals are all reduced a further 50% from the sale price. Lenny buys a pair of sandals from the bargain bin. The clerk notes that Lenny has saved \$9 from the list price of the sandals. How much did Lenny pay for the pair of sandals?

   (a) \$3 (b) \$4

   (c) \$5 (d) \$6

   (e) None of these

Solution. **(d)** Let L denote the list price of the sandals. Then L satisfies the equation

$$L(.8)(.5) = L - 9.$$

Solving, we get $L = 15$. Since Lenny saved \$9, he paid \$6.

3. What is the sum of the natural numbers less than or equal to 1000 that are divisible by 5 but not 25.

(a) 60,000 (b) 80,000

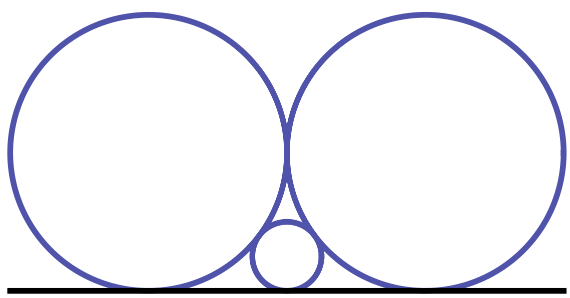
(c) 100,000 (d) 120,000

(e) None of these

Solution. **(b)**

$$\begin{aligned} \sum_{n=1}^{200} 5n - \sum_{m=1}^{40} 25n &= 5 \cdot \frac{200 \cdot 201}{2} - 25 \cdot \frac{40 \cdot 41}{2} \\ &= 80000. \end{aligned}$$

4. In the figure below, the two larger circles have radius equal to 1. Furthermore, all three circles are mutually tangent and tangent to the horizontal line. What is the radius of the smaller circle?

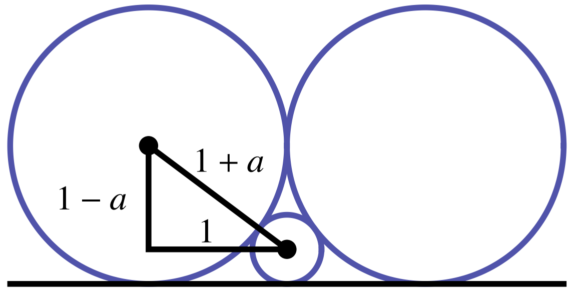


(a) $\frac{\sqrt{2}}{8}$ (b) $\frac{1}{5}$

(c) $\frac{1}{4}$ (d) $\frac{\sqrt{2}-1}{2}$

(e) None of these

Solution. **(c)** Draw the indicated right triangle, with respective leg lengths $1 - a$ and 1 and hypotenuse length $1 + a$.



From the Pythagorean Theorem, we have

$$1^2 + (1 - a)^2 = (1 + a)^2.$$

It is now easy to solve this equation, to obtain $a = \dfrac{1}{4}$.

5. How many 4-digit numbers in the form $a1b2$ are divisible by both 3 and 8, but not 9? (Here, a and b are digits, $a \neq 0$.)

(a) 1

(b) 2

(c) 4

(d) 8

(e) None of these

Solution. **(e)** Since $a \cdot 10^3$ is divisible by 8, we must have that the 3-digit number $1b2$ is divisible by 8.

$$\begin{aligned} 1b2 &= 100 + 10b + 2 \\ &\equiv 6 + 2b \bmod 8. \end{aligned}$$

This gives us the possibilities $b = 1, 5, 9$.

Also, by the divisibility rules for 3 and 9, the digit sum of $a1b2$, $a + b + 3$, is divisible by 3, but not 9. Thus, $a + b \equiv 0$ or $3 \bmod 9$. The only pairs of digits (a, b) that satisfy these restrictions are $(a, b) = (2, 1), (4, 5), (6, 9), (7, 5), (8, 1), (9, 9)$, giving 6 solutions, none of the answer choices provided.

6. Suppose the two parabolas whose respective equations are $y = ax^2 + bx + c$ and $x = ay^2 + by + c$ share the same vertex. Then...

(a) $b^2 + 2b = 4ac$

(b) $b^2 - b = 4ac$

(c) $b^2 - 2b = 4ac$

(d) $b^2 + b = 4ac$

(e) None of these

Solution. **(c)** By completing the square,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

we can see that the vertex of the parabola $y = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

Similarly, the vertex of the parabola $x = ay^2 + by + c$ is

$$\left(\frac{4ac - b^2}{4a}, -\frac{b}{2a}\right).$$

Thus, we must have

$$-\frac{b}{2a} = \frac{4ac - b^2}{4a},$$

which simplifies to $b^2 - 2b = 4ac$.

7. If

$$2010^{1776^x} = 1776^{2010},$$

then $x = \ldots$

(a) $(\log_{1776} 2010) + (\log_{1776} \log_{2010} 1776)$
(b) $(\log_{1776} 2010)\,(\log_{1776} \log_{2010} 1776)$
(c) $(\log_{1776} 2010) + (\log_{2010} \log_{1776} 2010)$
(d) $(\log_{1776} 2010)\,(\log_{2010} \log_{1776} 2010)$
(e) None of these

Solution. (a)

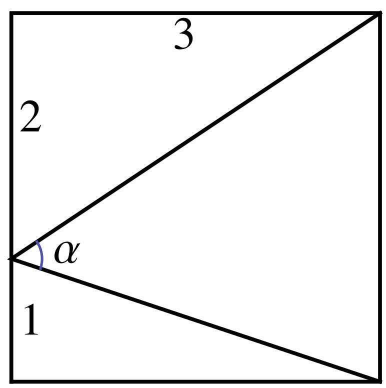
$$2010^{1776^x} = 1776^{2010} \implies 1776^x = \log_{2010} 1776^{2010}$$

$$\implies x = \log_{1776} \log_{2010} 1776^{2010}$$

$$\implies x = \log_{1776} (2010 \log_{2010} 1776)$$

$$\implies x = (\log_{1776} 2010) + (\log_{1776} \log_{2010} 1776)\,.$$

8. In the 3×3 square figure below, the vertex of the angle cuts the side in the indicated ratio of 2:1. Then, $\sin\alpha = \ldots$

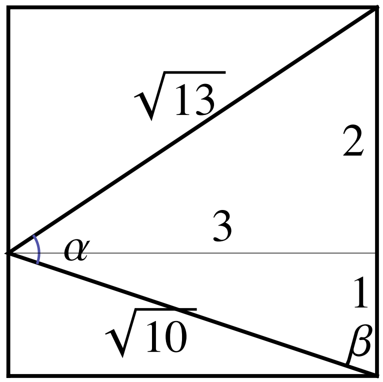


(a) $\dfrac{11}{\sqrt{130}}$ (b) $\dfrac{9}{\sqrt{130}}$

(c) $\dfrac{7}{\sqrt{130}}$ (d) $\dfrac{13}{\sqrt{130}}$

(e) None of these

Solution. (b) Draw the indicated horizontal line and with the help of the pythagorean theorem, compute the diagonal lengths.



By the Law of Sines,

$$\frac{\sin\alpha}{3} = \frac{\sin\beta}{\sqrt{13}}.$$

But then $\sin\beta = \dfrac{3}{\sqrt{10}}$, so we can solve for $\sin\alpha$,

$$\sin\alpha = \frac{9}{\sqrt{130}}.$$

9. What is the largest natural number n for which 2^n divides 100! ($100! = 100 \cdot 99 \cdot 98 \cdots 1$).

(a) 94

(b) 95

(c) 96

(d) 97

(e) None of these

Solution. (d) We need only count the number of multiples of 2, 4, 8, 16, 32 and 64 in the numbers 1 to 100.

- There are 50 multiples of 2,
- 25 multiples of 4,
- 12 multiples of 8,
- 6 multiples of 16,
- 3 multiples of 32 and

- 1 multiple of 64.

Summing these, gives $50 + 25 + 12 + 6 + 3 + 1 = 97$ as the answer.

10. How many ordered triples of positive integers, (x, y, z), are there that satisfy the equation $x + y + z = 2010$?

(a) 2,017,036

(b) 2,015,028

(c) 2,019,045

(d) 2,021,055

(e) None of these

Solution. **(a)** First note that the number of ordered pairs (x, y) that satisfy the equation $x + y = w$, where w is a fixed positive number bigger than 1, is $w - 1$. Now, write the equation $x + y + z = 2010$ as $x + y = 2010 - z$. Then for each value of $z = 1, \ldots, 2008$, there are $2010 - z - 1 = 2009 - z$ ordered pairs (x, y). So, the number of triples (x, y, z) is equal to the sum

$$\begin{aligned}\sum_{z=1}^{2008}(2009 - z) &= \sum_{i=1}^{2008} i \\ &= \frac{2008 \cdot 2009}{2} \\ &= 1004 \cdot 2009 \\ &= 2017036.\end{aligned}$$

2010
Leap Frog Relay Grades 9-12
Part II Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. How many distinct (no two the same) real roots does the polynomial $x^{2010} - x^{1776}$ have?

   (a) 3 (b) 2

   (c) 1 (d) 0

   (e) None of these

   Solution. (a) Factor the polynomial,

$$\begin{aligned} x^{2010} - x^{1776} &= x^{1776}\left(x^{234} - 1\right) \\ &= x^{1776}\left(x^{117} - 1\right)\left(x^{117} + 1\right). \end{aligned}$$

   So we see $x^{2010} - x^{1776}$ has only the three real roots 0, 1 and -1.

2. When a certain number N is divided by 15, the remainder is 12. What is the largest possible remainder if the same number N is divided by 12?

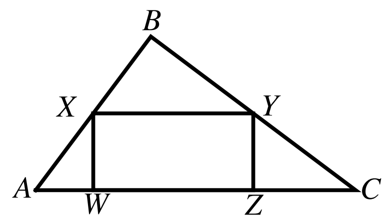
   (a) 2 (b) 3

   (c) 6 (d) 9

   (e) None of these

Solution. **(d)** The remainder fact means $N = 15q + 12$. And since $15 = 12 + 3$, we get $N = 12(q + 1) + 3q$. So when we divide N by 12, the remainder will be a multiple of 3. The largest multiple of 3 less than 12 is 9 which occurs if $q = 3$.

3. The rectangle $XYZW$ is inscribed in the triangle ABC so that X and Y are the respective midpoints of sides $\overline{AB}$ and $\overline{BC}$. (See figure below.)



Assuming that the side lengths (in inches) of the triangle are $AB = 3$, $BC = 4$ and $AC = 5$, then the area enclosed by the rectangle $XYZW$ is ....

(a) $\sqrt{15}$ in^2

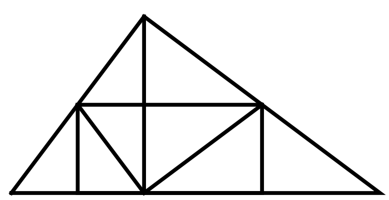
(b) $\sqrt{10}$ in^2

(c) $\frac{14}{3}$ in^2

(d) 3 in^2

(e) None of these

Solution. **(d)** The area enclosed by the rectangle is half of the area enclosed by the triangle. This is true in general, as you can see by the dissection pictured below.



Since $3^2 + 4^2 = 5^2$ we have, by the converse of the Pythagorean Theorem, $\triangle ABC$ is a right triangle, with right angle at $\angle B$. And so the area enclosed by $\triangle ABC$ is $\frac{1}{2}(3 \cdot 4) = 6$ in^2. And so, the area enclosed by the reactangle $XYZW$ is half of this, 3 in^2.

4. $\dfrac{1}{\log_2 2010} + \dfrac{1}{\log_3 2010} + \dfrac{1}{\log_5 2010} + \dfrac{1}{\log_{67} 2010} = \ldots$

(a) $\dfrac{1}{\log_{77} 2010}$ (b) $\dfrac{1}{\log_{2010} 77}$

(c) $\log_{77} 2010$ (d) $\log_{2010} 77$

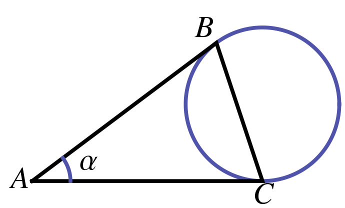
(e) None of these

Solution. **(e)** Use the identity $\log_b a = \dfrac{1}{\log_a b}$ to rewrite the sum as

$$\begin{aligned} \log_{2010} 2 + \log_{2010} 3 + \log_{2010} 5 + \log_{2010} 67 &= \log_{2010}(2 \cdot 3 \cdot 5 \cdot 67) \\ &= \log_{2010} 2010 \\ &= 1, \end{aligned}$$

none of the answer choices provided.

5. In the figure below, the segments $\overline{AB}$ and $\overline{AC}$ are tangent to the unit circle (radius = 1) at points B and C respectively. The measure of the angle at A is α. Then, the length of the chord BC is ...

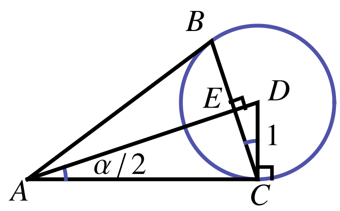


(a) $\cos(\alpha)$ (b) $2\cos(\alpha)$

(c) $2\cos(\alpha/2)$ (d) $\cos(\alpha/2)$

(e) None of these

Solution. **(c)** Let D be the center of the circle and consider the figure below. We can see $\triangle EDC \sim \triangle CDA$ by AA-triangle similarity. And consequently, $m\angle ECD = m\angle CAD = \alpha/2$.



Now, note that $EC = \cos\angle ECD$, and so

$$\begin{aligned} BC &= 2EC \\ &= 2\cos\angle ECD \\ &= 2\cos\alpha/2. \end{aligned}$$

6. Let $f(x) = \dfrac{x}{5} + \dfrac{5}{x}$. How many real numbers x satisfy the equation $f(f(x)) = f(x)$?

(a) 1

(b) 2

(c) 3

(d) 4

(e) None of these

Solution. **(d)** Let $y = f(x)$ and first solve the equation $f(y) = y$.

$$\begin{aligned} f(y) = y &\iff \frac{y}{5} + \frac{5}{y} = y \\ &\iff y^2 + 25 = 5y^2 \\ &\iff y^2 = \frac{25}{4} \\ &\iff y = \pm\frac{5}{2}. \end{aligned}$$

Now, solve $f(x) = \pm\dfrac{5}{2}$,

$$\begin{aligned} \frac{x}{5} + \frac{5}{x} = \pm\frac{5}{2} &\iff 2x^2 + 50 = \pm 25x \\ &\iff 2x^2 \mp 25x + 50 = 0 \\ &\iff (2x \mp 5)(x \mp 10) = 0 \\ &\iff x = \pm\frac{5}{2}, \pm 10, \end{aligned}$$

four solutions.

7. The infinite repeating base b number $(0.ddd\ldots)_b$ is equal to the real number $\frac{1}{5}$. Assuming that d is a non-zero digit (1, ..., 9), then the largest possible value for b is ...

(a) 44

(b) 45

(c) 46

(d) 47

(e) None of these

Solution. (c)

$$\begin{aligned}(0.ddd\ldots)_b &= \sum_{k=1}^{\infty} \frac{d}{b^k} \\ &= \frac{d}{1-1/b} - d \\ &= \frac{d}{b-1}.\end{aligned}$$

And so, we have the equality

$$\frac{d}{b-1} = \frac{1}{5},$$

which is equivalent to

$$5d + 1 = b.$$

Thus, the largest possible value for b occurs when $d = 9$, giving $b = 5 \cdot 9 + 1 = 46$.

8. The real number $\sqrt{2010} - \sqrt{1776}$ is the root of a fourth degree polynomial in the form $p(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, where c_3, c_2, c_1, c_0 are integers. Then, the sum of the roots of $p(x)$ is ...

(a) -468

(b) -234

(c) 0

(d) 234

(e) None of these

Solution. **(c)** Suppose $x = \sqrt{a} - \sqrt{b}$. Rewrite this as $x + \sqrt{b} = \sqrt{a}$ and square both sides, getting $x^2 + 2\sqrt{b}x + b = a$. Rearrange terms, $2\sqrt{b}x = (a-b) - x^2$, and square again, $4bx^2 = (a-b)^2 - 2(a-b)x^2 + x^4$. Rearrange one more time to get

$$p(x) = x^4 - 2(a+b)x^2 + (a-b)^2.$$

If r_1, r_2, r_3, r_4 are the roots of $p(x)$, then $p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$. Upon expanding, we see that the coefficient of x^3 is $c_3 = -(r_1 + r_2 + r_3 + r_4)$. Since $c_3 = 0$, we get that the sum of the roots is

$$r_1 + r_2 + r_3 + r_4 = 0.$$

9. $\csc(2010^\circ) = \ldots$.

(a) $\dfrac{1}{\sqrt{2}}$ (b) $-\dfrac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) $-\sqrt{2}$

(e) None of these

Solution. **(e)** Divide 2010 by 360 to get $2010 = 5 \cdot 360 + 210$. Then,

$$\begin{aligned}
\csc(2010^\circ) &= \csc(210^\circ) \\
&= \frac{1}{\sin(210^\circ)} \\
&= \frac{1}{\sin(180^\circ + 30^\circ)} \\
&= \frac{1}{-\sin(30^\circ)} \\
&= \frac{1}{-1/2} \\
&= -2,
\end{aligned}$$

none of the answer choices provided.

10. Suppose $x = \sqrt{2}^{\sqrt{2}}$. Then

$$2^x - (x^x)^{\sqrt{2}} = \dots$$

(a) $-\sqrt{2}$

(b) $\sqrt{2}$

(c) $\dfrac{1}{\sqrt{2}}$

(d) 0

(e) None of these

Solution. (d) First note that

$$\begin{aligned} x &= \sqrt{2}^{\sqrt{2}} \\ &= 2^{\sqrt{2}/2} \\ &= 2^{1/\sqrt{2}}. \end{aligned}$$

So,

$$\begin{aligned} x\sqrt{2} &= 2^{1\sqrt{2}} \cdot 2^{1/2} \\ &= 2^{\frac{1}{\sqrt{2}}+\frac{1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} (x^x)^{\sqrt{2}} &= x^{(x\sqrt{2})} \\ &= \left(2^{\frac{1}{\sqrt{2}}}\right)^{\left(2^{\frac{1}{\sqrt{2}}+\frac{1}{2}}\right)} \\ &= \left(2^{2^{-\frac{1}{2}}}\right)^{\left(2^{\frac{1}{\sqrt{2}}+\frac{1}{2}}\right)} \\ &= 2^{\left(2^{\left(-\frac{1}{2}\right)} \cdot 2^{\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)}\right)} \end{aligned}$$

$$= \quad 2^{\left(2^{\frac{1}{\sqrt{2}}}\right)}$$

$$= \quad 2^x.$$

This gives us the simplification,

$$2^x - (x^x)^{\sqrt{2}} = 0.$$