CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART I

April 22^{nd}, 2017

1. $2017(2017^{2017}) =$

(a) 2017^{2018}
(b) 4034^{2017}
(c) 2017^{4034}
(d) 4068289^{2017}
(e) $2017^{4068289}$

2. Each day Jane ate 25% of the candies that were in her jar at the beginning of that day. At the end of the second day, 36 candies remained. How many candies were in the jar originally?

   (a) 48
   (b) 56
   (c) 64
   (d) 72
   (e) 80

3. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

(a) 0
(b) 4
(c) 6
(d) 7
(e) 9

4. If $|x-2| = p$, where $x < 2$, then $x - p =$

(a) -2
(b) 2
(c) $2-2p$
(d) $2p-2$
(e) $|2p-2|$

5. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

(a) 21
(b) 60
(c) 119
(d) 180
(e) 231

6. How many positive integers b have the property that $\log_b 729$ is a positive integer?

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

7. Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 non overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in figure 100?

Figure 0

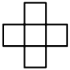


Figure 1

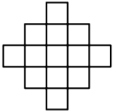


Figure 2

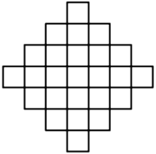


Figure 3

(a) 10401
(b) 19801
(c) 20201
(d) 39801
(e) 40801

8. Mrs. Rodriguez gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which calculated the class average after each score was entered. Mrs. Rodriguez noticed that after each score was entered, the average was always an integer. The scores listed in ascending order were 71, 76, 80, 82, and 91. What was the last score Mrs. Rodriguez entered?

(a) 71
(b) 76
(c) 80
(d) 82
(e) 91

9. The point $P(1, 2, 3)$ is reflected in the xy-plane, then its image Q is rotated by 180° about the x-axis to produce R, and finally, R is translated by 5 units in the positive y-direction to produce S. What are the coordinates of S?

(a) $(1, 7, -3)$
(b) $(-1, 7, -3)$
(c) $(-1, -2, 8)$
(d) $(-1, 3, 3)$
(e) $(1, 3, 3)$

10. Two non-zero real numbers a, and b satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

(a) -2
(b) $\frac{-1}{2}$
(c) $\frac{1}{3}$
(d) $\frac{1}{2}$
(e) 2

11. Let M, F, and D be non-negative integers such that $M + F + D = 12$. What is the maximum value of $M \cdot F \cdot D + M \cdot F + F \cdot D + M \cdot D$?

(a) 62
(b) 72
(c) 92
(d) 102
(e) 112

12. One morning each member of John's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. John drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7

13. When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

(a) 3
(b) 6
(c) 9
(d) 17
(e) 20

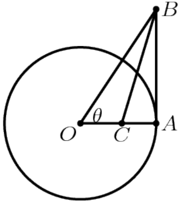
14. Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(a) $-1/3$
(b) $-1/9$
(c) 0
(d) $5/9$
(e) $5/3$

15. A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered $1, 2, \ldots 17$, the second row is numbered $18, 19, \ldots, 34$ and so on down the board. If the board is renumbered so that the left column, top to bottom, is $1, 2, \ldots, 13$, the second column is $14, 15, \ldots, 26$ and so on across the board, some squares have the same numbers in both numbering systems. Find the sum of the numbers in these squares (under either system).

(a) 222
(b) 333
(c) 444
(d) 555
(e) 666

16. A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies of $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then $OC =$



(a) $\sec^2\theta - \tan\theta$

(b) $\frac{1}{2}$

(c) $\frac{\cos^2\theta}{1+\sin\theta}$

(d) $\frac{1}{1+\sin\theta}$

(e) $\frac{\sin\theta}{\cos^2\theta}$

17. In year N, the 300^{th} day of the year is a Tuesday. In year $N+1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100^{th} day of the year $N-1$ occur?

(a) Thursday
(b) Friday
(c) Saturday
(d) Sunday
(e) Monday

18. In triangle ABC, $AB = 13, BC = 14, AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of the triangle ADE?

(a) 2
(b) 2.5
(c) 3
(d) 3.5
(e) 4

19. If x, y, and z are positive numbers satisfying

$$x + 1/y = 4, \quad y + 1/z = 1, \text{ and } z + 1/x = 7/3$$

Then what is the value of xyz?

(a) 2/3
(b) 1
(c) 4/3
(d) 2
(e) 7/3

20. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

(a) $\frac{1}{2m+1}$

(b) m

(c) $1-m$

(d) $\frac{1}{4m}$

(e) $\frac{1}{8m^2}$

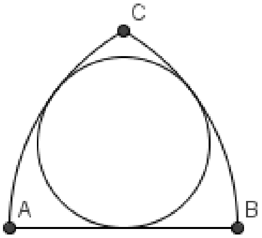
21. You write five letters to different people, and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes, with one letter in each envelope, so that none of them is in the correct envelope?

(a) 36
(b) 40
(c) 42
(d) 44
(e) 52

22. Professor Gamble buys a lottery ticket, which requires that he pick six different integers from 1 through 46, both inclusive. He chooses his numbers so that the sum of the base-ten logarithms of his six numbers is an integer. It so happens that the integers on the winning ticket have the same property – the sum of the base-ten logarithms is an integer. What is the probability that Professor Gamble holds the winning ticket?

(a) 1/5
(b) 1/4
(c) 1/3
(d) 1/2
(e) 1

23. If circular arcs AC and BC have centers B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$. If the length of $\overset{\frown}{BC}$ is 12, then the circumference of the circle is



(a) 24
(b) 25
(c) 26
(d) 27
(e) 28

24. How many ordered 4-tuples of positive integers (a, b, c, d) satisfy $a + b + c + d \leq 15$?

(a) 1252
(b) 1304
(c) 1365
(d) 1524
(e) none of the above

25. Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look like the other.)

(a) 210
(b) 560
(c) 840
(d) 1260
(e) 1680

26. The sum of the first n positive integers is a three-digit number in which all of the digits are the same. What is the sum of the digits of n?

(a) 6
(b) 9
(c) 12
(d) 15
(e) 18

27. Consider the sequence of numbers: $4, 7, 1, 8, 9, 7, 6, \ldots$. For $n > 2$, the n^{th} term of the sequence is the units digit of the sum of the previous two terms. Let S_n denote the sum of the first n terms of this sequence. The smallest value of n for which $S_n > 10,000$ is:

(a) 1992
(b) 1999
(c) 2001
(d) 2002
(e) 2004

28. On the island of Knights and Knaves there live only two types of people: Knights (who always speak the truth) and Knaves (who always lie). I met two men who lived there and asked the taller man if they were both Knights. He replied, but I could not figure out what they were, so I asked the shorter man if the taller was a Knight. He replied, and after that I knew which type they were. Were the men Knights or Knaves?

(a) They were both Knights.
(b) They were both Knaves.
(c) The taller was a Knight and the shorter was a Knave.
(d) The taller was a Knave and the shorter was a Knight.
(e) Not enough information is given.

29. Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

(a) 3
(b) 4
(c) 5
(d) 8
(e) 9

30. Several sets of prime numbers, such as $\{7, 83, 421, 659\}$ use each of the nine non-zero digits exactly once. What is the smallest possible sum such a set of primes could have?

(a) 193
(b) 207
(c) 225
(d) 252
(e) 447

Solutions

1. A
2. C
3. C
4. C
5. C
6. E
7. C
8. C
9. E
10. E
11. E
12. C
13. E
14. B
15. D
16. D

17. A
18. C
19. B
20. D
21. D
22. B
23. D
24. C
25. E
26. B
27. B
28. D
29. C
30. B

CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART II

April 22^{nd}, 2017

1. Sandwiches at Jill's Fast Food cost \$3 each and sodas cost \$2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

   (a) 31
   (b) 32
   (c) 33
   (d) 34
   (e) 35

2. Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

(a) $-h$
(b) 0
(c) h
(d) $2h$
(e) h^3

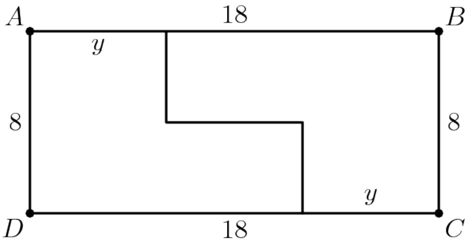
3. The ratio of Andrew's age to Michael's age is $3 : 5$. Michael is 30 years old. How old is Andrew?

(a) 15
(b) 18
(c) 20
(d) 24
(e) 50

4. A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

(a) 17
(b) 19
(c) 21
(d) 22
(e) 23

5. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



(a) 6
(b) 7
(c) 8
(d) 9
(e) 10

6. Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half the pizza. The cost of a plain pizza was 8 dollars, and there was an additional cost of 2 dollars for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each paid for what he had eaten. How many more dollars did Dave pay than Doug?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

7. Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?

(a) 7
(b) 8
(c) 9
(d) 10
(e) 11

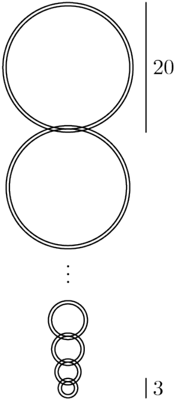
8. How many sets of two or more consecutive positive integers have a sum of 15?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

9. Oscar buys 13 pencils and 3 erasers for \$1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?

(a) 10
(b) 12
(c) 15
(d) 18
(e) 20

10. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



(a) 171

(b) 173

(c) 182

(d) 188

(e) 210

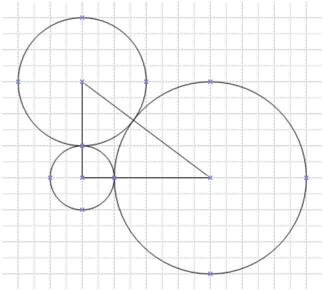
11. For how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

(a) 3
(b) 6
(c) 9
(d) 10
(e) 11

12. Which of the following describes the graph of $(x + y)^2 = x^2 + y^2$?

(a) the empty set
(b) one point
(c) two lines
(d) a circle
(e) the entire plane

13. The vertices of a 3-4-5 right triangle are centers of three mutually externally tangent circles as shown below. What is the sum of the areas of the three circles?



(a) 12π

(b) $25\pi/2$

(c) 13π

(d) $27\pi/2$

(e) 14π

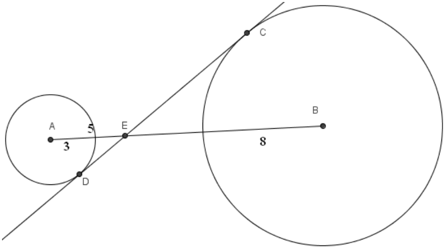
14. Two farmers agree that pigs are worth 300 dollars and goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of pigs or goats as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

(a) 5
(b) 10
(c) 30
(d) 90
(e) 210

15. Suppose $\cos x = 0$ and $\cos(x+z) = 1/2$. What is the smallest possible positive value of z?

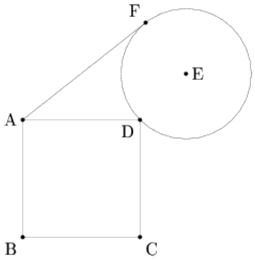
(a) $\pi/6$
(b) $\pi/3$
(c) $\pi/2$
(d) $5\pi/6$
(e) $7\pi/6$

16. Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent intersects the circles at D and C, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD?



(a) 13
(b) $44/3$
(c) $\sqrt{221}$
(d) $\sqrt{255}$
(e) $55/3$

17. Square $ABCD$ has side length s, a circle centered at E has radius r, and r and s are both rational. The circle passes through D, and D lies on $\overline{BE}$. Point F lies on the circle, on the same side of $\overline{BE}$ as A. Segment AF is tangent to the circle, and $AF = \sqrt{9+5\sqrt{2}}$. What is r/s?



(a) $1/2$
(b) $5/9$
(c) $3/5$
(d) $5/3$
(e) $9/5$

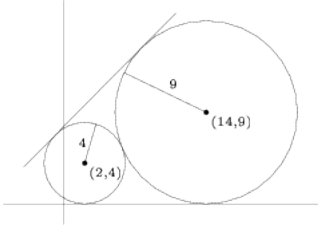
18. The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f(1/x) = x$$

What is the largest set of real numbers that can be in the domain of f?

(a) $\{x | x \neq 0\}$
(b) $\{x | x < 0\}$
(c) $\{x | x > 0\}$
(d) $\{x | x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$
(e) $\{-1, 1\}$

19. Circles with centers $(2,4)$ and $(14,9)$ have radii 4 and 9, respectively. The equation of a common external tangent to the circles can be written in the form $y = mx + b$ with $m > 0$. What is b?



(a) 908/119
(b) 909/119
(c) 130/17
(d) 911/119
(e) 912/119

20. Let

$$S_1 = \{(x, y) | \log_{10}(1 + x^2 + y^2) \leq 1 + \log_{10}(x + y)\}$$

and

$$S_2 = \{(x, y) | \log_{10}(2 + x^2 + y^2) \leq 2 + \log_{10}(x + y)\}$$

What is the ratio of the area of S_2 to the area of S_1?

(a) 98
(b) 99
(c) 100
(d) 101
(e) 102

21. A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug would have visited each of the vertices exactly once?

(a) 1/2187
(b) 1/729
(c) 2/243
(d) 1/81
(e) 5/243

22. Kim's flight took off from Newark at 10:34 AM and landed in Miami at 1:18 PM. Both cities are in the same time zone. If her flight took h hours and m minutes, with $0 < m < 60$, what is $h + m$?

(a) 46
(b) 47
(c) 50
(d) 53
(e) 54

23. Which of the following is equal to $1 + \dfrac{1}{1 + \frac{1}{1+1}}$?

(a) $5/4$
(b) $3/2$
(c) $5/3$
(d) 2
(e) 3

24. Four coins are picked out of a piggy bank that contains a collection of pennies, nickels, dimes, and quarters. Which of the following could not be the total value of the four coins, in cents?

(a) 15
(b) 25
(c) 35
(d) 45
(e) 55

25. Eric plans to compete in a triathlon. He can average 2 miles per hour in the 1/4 mile swim and 6 miles per hour in the 3 mile run. His goal is to finish the triathlon in 2 hours. To accomplish his goal what must his average speed in miles per hour be for the 15 mile bicycle ride?

(a) 120/11
(b) 11
(c) 56/5
(d) 45/4
(e) 12

26. What is the sum of the digits of the square of 111,111,111?

(a) 18
(b) 27
(c) 45
(d) 63
(e) 81

27. A carton contains milk that contains 2% fat, an amount that is 40% less fat than the amount contained in a carton of whole milk. What is the percentage of fat in whole milk?

(a) 12/5
(b) 3
(c) 10/3
(d) 38
(e) 42

28. Three generations of the Wen family are going to the movies, two from each generation. The two members of the youngest generation receive 50% discount as children. The two members of the oldest generation receive a 25% discount as senior citizens. The two members of the middle generation receive no discount. Grandfather Wen, whose senior ticket costs \$6.00, is paying for everyone. How many dollars must he pay?

(a) 34
(b) 36
(c) 42
(d) 46
(e) 48

29. Positive integers a, b, and 2009, with $a < b < 2009$, form a geometric sequence with an integer ratio. What is a?

(a) 7
(b) 41
(c) 49
(d) 289
(e) 2009

30. Triangle ABC is right angled at B. Point D is the foot of the altitude drawn from B, $AD = 3$, and $DC = 4$. What is the area of triangle ABC?

(a) $4\sqrt{3}$
(b) $7\sqrt{3}$
(c) 21
(d) $14\sqrt{3}$
(e) 42

Solutions

1. A
2. C
3. B
4. E
5. A
6. D
7. B
8. C
9. A
10. B
11. E
12. C
13. E
14. C
15. A
16. B

17. B
18. E
19. E
20. E
21. C
22. A
23. C
24. A
25. A
26. E
27. C
28. B
29. B
30. B