2017
Leap Frog Relay Grades 9-10
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. How many (distinct) real number solutions are there to the equation:

$$x^{2021} - 4x^{2019} = x^{2019} - 4x^{2017}$$

(a) 2 (b) 3

(c) 4 (d) 5

(e) None of these

Solution. (d) Factoring both sides of the equation gives:

$$x^{2019}(x^2 - 4) = x^{2017}(x^2 - 4)$$

The choice $x = 0$ is a solution. Suppose $x \neq 0$, we may divide both sides by x^{2017} and get

$$x^2(x^2 - 4) = x^2 - 4$$

If $x = \pm 2$, we have two more solutions. Suppose $x \neq \pm 2$, we may divide both sides by $(x^2 - 4)$ and obtain

$$x^2 = 1$$

which has two solutions $x = \pm 1$. We have 5 total solutions, $0, \pm 1, \pm 2$.

2. A palindromic number is a number that is exactly the same if the order is reversed; for example, 1225221 or 1331. How many four-digit palindromic numbers are there whose sums of digits are divisible by four?

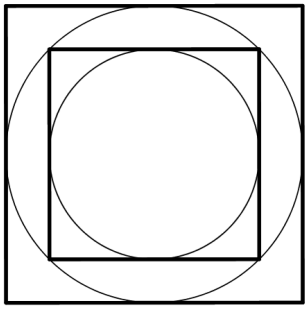
(a) 20 (b) 45

(c) 50 (d) 100

(e) None of these

Solution. **(b)** For a four-digit palindromic number $abba$, there are nine possible choices for a: $1, 2, \ldots, 9$; zero will make the number a three digit number. The sum of digits is $2a + 2b = 2(a + b)$, and this sum is divisible by 4 if $a + b$ is divisible by 2. For a even, there are four possibilities a, $2, 4, 6, 8$. To be divisible by two, b must be even as well, with five possible choices for b, $0, 2, 4, 6, 8$. This totals to 20 possible choices for a even. For a odd, there are five possibilities $1, 3, 5, 7, 9$ for the value of a. To be divisible by 2, b must also be odd, with the same possible choices. This totals to 25 possible choices for a odd. Therefore, there are 45 possible such palindromic numbers.

3. A circle is inscribed in a square. A square is inscribed in that circle. A second circle is inscribed in that square. What is the ratio of the area of the smallest circle to the area of the largest square?



(a) $\pi/2$ (b) $\pi^2/4$

(c) $\pi/8$ (d) $\pi^2/16$

(e) None of these

Solution. (c) If a circle inscribed in a square has radius r, then the square has side length $2r$. The respective areas are πr^2 and $4r^2$ and the ratio of the area of the circle to that of the square is $\pi/4$. For a square inscribed into a circle of radius r, the diagonal of the square is a diameter length $2r$ and the side length of the square is $\sqrt{2}r$. The areas are $2r^2$ and πr^2 with a ratio from the square to circle of $2/\pi$. The ratio of the area of the smallest circle to the area of the largest square is the product

$$\frac{\pi}{4} \cdot \frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{\pi}{8}.$$

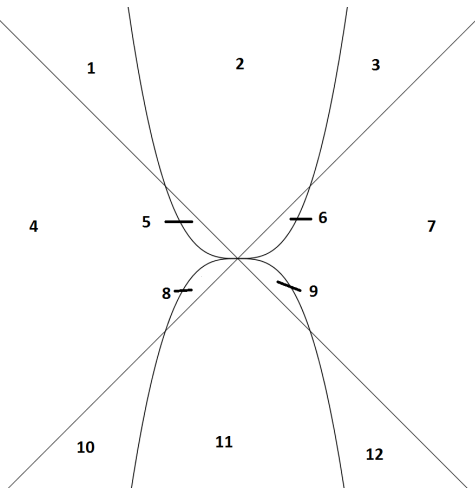
4. The graphs of $y^2 = x^6$ and $|y| = |x|$ are plotted on the Cartesian plane. How many different regions do these graphs divide the plane into?

(a) 8 (b) 12

(c) 14 (d) 16

(e) None of these

Solution. (b)



5. An urn contains ten balls, either red or blue. You may take balls from the urn one at a time and you do not place them back into the urn. You must take at least seven balls to guarantee to take at least three red balls. How many blue balls are in the urn at the start?

   (a) 3 (b) 4

   (c) 5 (d) 6

   (e) None of these

   Solution. (b) If you must take seven balls to obtain at least three red balls, then the other four balls must be blue, and these must be all of the blue balls in the urn.

6. The average wage at a local restaurant is \$10 an hour among 24 employees. A new manager is hired and the new average wage is \$10.40 an hour among the total 25 employees. How much does the new manager make per hour?

   (a) 20 (b) 22

   (c) 24 (d) 26

   (e) None of these

   Solution. (a) The total per hour wages for twenty four employees is \$240. If the average for 25 employees is \$10.40, then the total is \$260. The difference is the new manager's wage of \$20 per hour.

7. A parabola $y = ax^2 + bx + c$ has its vertex at $(6, 15)$ and contains the point $(0, -3)$. What is the product abc?

   (a) $-\frac{4}{9}$ (b) -1

   (c) 4 (d) 9

   (e) None of these

Solution. **(d)** Substituting $(0, -3)$ into the equation tells us that $c = -3$. The x-coordinate of the vertex is $-b/2a = 6$ and thus $b = -12a$. Substituting this and the vertex $(6, 15)$ into the equation for the parabola, we have

$$\begin{aligned} 15 &= a \cdot 6^2 - 12a \cdot 6 - 3 \\ 18 &= -36a \\ a &= -\frac{1}{2} \\ b &= 6 \quad (\text{since } b = -12a) \end{aligned}$$

Therefore, the product abc is: $abc = (-\frac{1}{2}) \cdot (6) \cdot (-3) = 9$.

8. The value of x is 40% of y. What is y as a percent of x?

   (a) 60%
   (b) 140%
   (c) 250%
   (d) 400%
   (e) None of these

   Solution. **(c)** Since $40\% = \dfrac{40}{100} = \dfrac{2}{5}$, then the inverse is $\dfrac{5}{2} = \dfrac{250}{100} = 250\%$. Hence, y is 250% of x.

9. The sum of eight consecutive integers is 212. What is the sum of the first and last integers?

   (a) 52
   (b) 53
   (c) 54
   (d) 55
   (e) None of these

   Solution. **(b)** The sum of $x, x+1, x+2, \ldots, x+7$ is $8x+28$. To solve for x, we solve $8x + 28 = 212$ and find $x = 23$. The first and last terms are 23 and 30, so their sum is 53.

10. If $4^{5x+3} = 8^{2x+1}$, then $16^x =$?

(a) $\frac{3}{4}$

(b) $\frac{-3}{4}$

(c) $\frac{1}{2}$

(d) $\frac{1}{8}$

(e) None of these

Solution. (d) The equation can be rewritten in base 2: $2^{10x+6} = 2^{6x+3}$. Exponential functions are one-to-one, so $10x + 6 = 6x + 3$. Solving for x, we have

$$10x + 6 = 6x + 3$$
$$4x = -3$$

Then $16^x = 2^{4x} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

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11. Find the real solution to

$$\frac{1}{x} + \frac{x}{5} = \frac{1+x}{x+5}$$

(a) $x = -\sqrt[3]{25}$

(b) $x = -\sqrt[3]{10}$

(c) $x = 1 - \sqrt[3]{10}$

(d) $x = 1 - \sqrt[3]{25}$

(e) None of these

Solution. (a)

$$\begin{aligned}
\frac{1}{x} + \frac{x}{5} &= \frac{1+x}{x+5} \\
\frac{5+x^2}{5x} &= \frac{1+x}{x+5} \\
(5+x^2)(x+5) &= (1+x)\cdot 5x \\
x^3 + 5x^2 + 5x + 25 &= 5x^2 + 5x \\
x^3 &= -25 \\
x &= -\sqrt[3]{25}
\end{aligned}$$

12. The lines $y = 2x + b$ and $y = x + 2017$ meet on a point on the line $y = 4x + 16$. Determine b.

(a) 1250 (b) 1251

(c) 1351 (d) 1350

(e) None of these

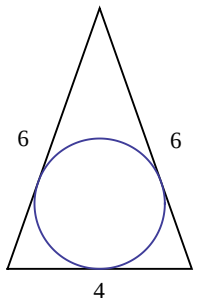
Solution. (d) We note that the three lines must have a common point, which is the point where the first two lines intersect. Set the second and third equations equal to each other. We have:

$$\begin{aligned} x + 2017 &= 4x + 16 \\ 2001 &= 3x \\ x &= 667. \end{aligned}$$

Set the first two equations equal to each other to solve for b:

$$\begin{aligned} 2x + b &= x + 2017 \\ b &= -x + 2017 \\ b &= -667 + 2017 \text{ (by plugging 667 into } x) \\ b &= 1350. \end{aligned}$$

13. A circle is inscribed in the isosceles triangle with respective side lengths 6, 6 and 4. Determine the area of the inscribed circle.

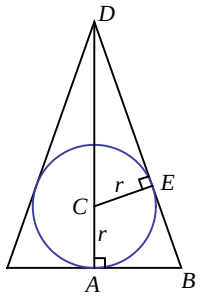


(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

(e) None of these

Solution. **(e)** Label the figure as indicated below.



We have $2 = AB = EB$, so $DE = 4$. Also, by the Pythagorean Theorem applied to the right triangle $\triangle DAB$, we have $DA = 4\sqrt{2}$. The pair of similar triangles $\triangle DEC \sim \triangle DAB$ implies proportional sides $DA/DE = AB/CE \implies 4\sqrt{2}/4 = 2/r \implies r = \sqrt{2}$. Thus, the area of the circle is $\pi r^2 = 2\pi$, none of the answer choices provided.

14. A music player has a list price of \$100. However, the store is having a 10% off sale for the month of April. But you are in luck, because you came on a Tuesday in April when the store gives an additional 15% *off the sale price* at the register. Assuming sales tax is 10% of the register price, how much are you going to pay for the music player?

(a) \$83.85 (b) \$83.95

(c) \$84.05 (d) \$84.15

(e) None of these

Solution. **(d)** The April sale price for the player is \$90. The register price will then be 85% of \$90, which is \$76.50. Sales tax (10%) on \$76.50 is \$7.65. So you will pay the sum of \$76.50 and \$7.65, which is \$84.15.

15. $a + b + c = -4$ and $ab + ac + bc = 3$. What is $a^2 + b^2 + c^2$?

(a) 10 (b) 16

(c) 19 (d) 25

(e) None of these

Solution. **(a)** We have that,

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.$$

Solving for $a^2+b^2+c^2$, we obtain:

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+ac+bc) = (-4)^2 - 6 = 10.$$

16. Suppose you take two steps to the right, four steps to the left, six steps to the right, and so on, where each number of steps is a multiple of two. Suppose the last move is 17422 steps to the right. How far are you from the starting position after the last step?

(a) 8710 steps to the left

(b) 8712 steps to the right

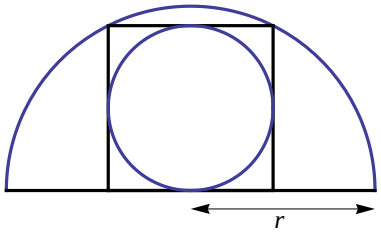
(c) 13065 steps to the left

(d) 17422 steps to the right

(e) None of these

Solution. **(b)** For every pair of moves, you end up two to the left. If the last move was 17422 steps to the right, then the number of pairs of steps is $17420/4 = 4355$, which means you should be 8710 steps to the left before taking the 17422 steps to the right. You should be a total 8712 steps to the left.

17. The semicircle pictured below has a radius of to r inches. The square is inscribed in the semicircle and the smaller circle is inscribed in the square. What is the area of the smaller circle in terms of r?



(a) $\dfrac{\pi r^2}{6}$ inches^2.

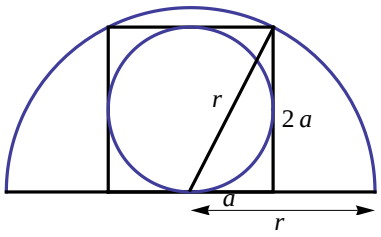
(b) $\dfrac{\pi r^2}{5}$ inches^2.

(c) $\dfrac{\pi r^2}{4}$ inches^2.

(d) $\dfrac{\pi r^2}{3}$ inches^2.

(e) None of these

Solution. (b) Let a be the radius of the smaller circle. We can see from the figure below that a, $2a$ and r are the two legs and, respectively, hypotenuse of a right triangle.



So, by the Pythagorean Theorem, we have:

$$a^2 + (2a)^2 = r^2 \Longrightarrow 5a^2 = r^2 \Longrightarrow a^2 = \frac{r^2}{5}.$$

Thus, the area of the smaller circle is:

$$\pi a^2 = \frac{\pi r^2}{5} \text{ inches}^2.$$

18. Four oranges weigh as much as three apples. Nine limes weight as much as two oranges. How many limes weight as much as seven apples weigh?

(a) 21 (b) 28

(c) 35 (d) 42

(e) None of these

Solution. (d) The amount is $7 \cdot \frac{4}{3} \cdot \frac{9}{2} = 42$.

19. Suppose $\log_2(\log_3(\log_2 x)) = 1$. Which of the following is true about x?

(a) $10 < x \leq 100$ (b) $100 < x \leq 1000$

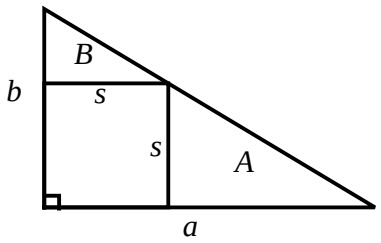
(c) $1000 < x \leq 10000$ (d) $x > 10000$

(e) None of these

Solution. (b)

$$\begin{aligned}\log_2(\log_3(\log_2 x)) &= 1\\ \log_3(\log_2 x) &= 2\\ \log_2 x &= 3^2 = 9\\ x &= 2^9 = 512\end{aligned}$$

20. In the figure below, the large right triangle has respective leg lengths a and b, as pictured. The s by s square is inscribed in the triangle. The respective areas of the two smaller right triangles are A and B as indicated. Determine the ratio of the areas A/B as a function of a and b.



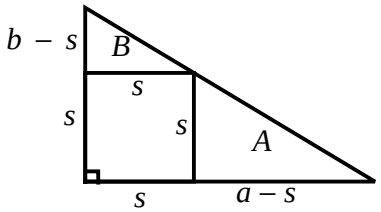
(a) $A/B = a^2/b^2$ (b) $A/B = a/b$

(c) $A/B = (ab)/(a+b)$ (d) $A/B = \sqrt{a^2+b^2}/(a+b)$

(e) None of these

Solution. (a) If we compare the similar triangle pair that is the small top triangle with the large triangle, we get equal ratios:

$$\frac{b-s}{s} = \frac{b}{a} \Longrightarrow s = \frac{ab}{a+b}.$$



So,

$$\begin{aligned}\frac{A}{B} &= \frac{\frac{1}{2}(a-s)s}{\frac{1}{2}(b-s)s} \\ &= \frac{a-s}{b-s} \\ &= \frac{a-\frac{ab}{a+b}}{b-\frac{ab}{a+b}} \\ &= \frac{a^2}{b^2}.\end{aligned}$$