2017
Leap Frog Relay Grades 11-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. If r_1 and r_2 are the two real number solutions to the equation $x^2 + x = 2017$, then $(r_1 + r_2)^{2017} =$ ________.

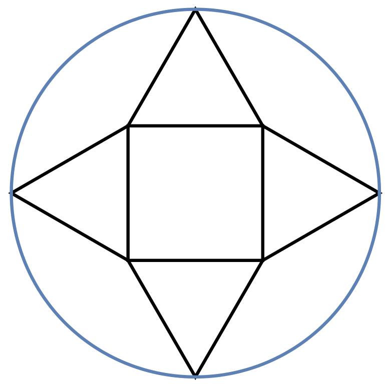
   (a) 0
   (b) 1
   (c) 2^{2017}
   (d) -2^{2017}
   (e) None of these

   Solution. (e) Rewriting as $x^2 + x - 2017 = 0$, and using the quadratic formula, we can see that the two solutions are

   $$\frac{-1 \pm \sqrt{1 - 4(-2017)}}{2} = \frac{-1 \pm \sqrt{8069}}{2}.$$

   Thus, $r_1 + r_2 = -1$ and so $(r_1 + r_2)^{2017} = -1$, none of the answer choices provided.

2. The central square is sharing its sides with 4 equilateral triangles, and the combined figure is inscribed in the circle as pictured below.



What is the ratio of circle area to square area?

(a) $\sqrt{6}\pi$

(b) $\pi\left(1+\frac{\sqrt{3}}{2}\right)$

(c) 2π

(d) $\pi\left(1+\sqrt{3}\right)$

(e) None of these

Solution. **(b)** Since we are computing the ratio of areas, we are allowed to scale the figure any way we please. So, choose the square side length to be 2, with area equal to 4. The height of each equilateral triangle is $\sqrt{3}$. Thus, the radius of the circle is $1+\sqrt{3}$. The ratio of circle area to square area is then

$$\frac{\pi(1+\sqrt{3})^2}{4} = \pi\left(1+\frac{\sqrt{3}}{2}\right).$$

3. If you triple the radius of a circle, then the resulting *percentage* increase in circle area is ________.

(a) 300%

(b) 600%

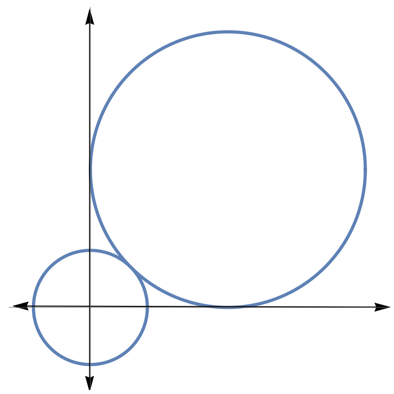
(c) 800%

(d) 900%

(e) None of these

Solution. **(c)** If r is the original radius, with circle area equal to πr^2, then the result of tripling the radius results in an area of $9\pi r^2$, an 800% increase.

4. In the figure below, the smaller circle is centered at the origin and has radius equal to a, while the larger circle is mutually tangent to the smaller circle and the two coordinate axes, with radius equal to b. Then, $b/a =$ ________.



(a) $\frac{3}{2}$

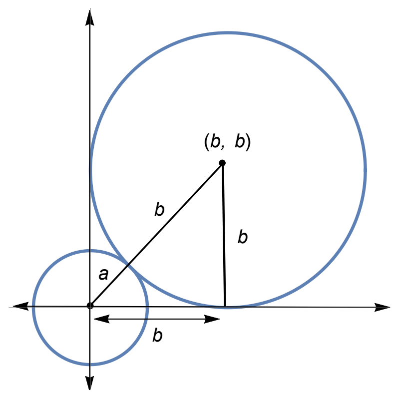
(b) 2

(c) $1 + \sqrt{2}$

(d) $\frac{5}{2}$

(e) None of these

Solution. (c) We label our figure as indicated below.



Using the Pythagorean Theorem, we get

$$b^2 + b^2 = (a+b)^2 \Longrightarrow \left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right) - 1 = 0.$$

This is a quadratic equation in b/a, with two solutions,

$$\frac{b}{a} = 1 \pm \sqrt{2}.$$

We choose the positive solution,

$$\frac{b}{a} = 1 + \sqrt{2}.$$

5. If $\log_{4034} 2 = a$, then $\log_{2017} 4034 =$ ______.

(a) $\dfrac{1}{a}$ (b) $\dfrac{1}{1+a}$

(c) $\dfrac{1}{2a}$ (d) $\dfrac{2}{1+a}$

(e) None of these

Solution. (e)

$$\begin{aligned} \log_{2017} 4034 &= \frac{1}{\log_{4034} 2017} \\ &= \frac{1}{\log_{4034} 4034 - \log_{4034} 2} \\ &= \frac{1}{1-a}, \end{aligned}$$

none of the answer choices provided.

6. If $\sqrt[3]{4} \cdot \sqrt[4]{x} = 2\sqrt[12]{32}$, then $x =$ ______.

(a) 64 (b) 8

(c) 4 (d) 32

(e) None of these

Solution. (b) Convert to fractional exponents to get

$$\sqrt[3]{4} \cdot \sqrt[4]{x} = 2\sqrt[12]{32} \Longleftrightarrow 2^{2/3} \cdot x^{1/4} = 2 \cdot 2^{5/12}.$$

First, solving for $x^{1/4}$,

$$\begin{aligned} x^{1/4} &= \frac{2 \cdot 2^{5/12}}{2^{2/3}} \\ &= 2^{3/4}. \end{aligned}$$

Thus, $x = 8$.

7. If $\sin(x+\pi) = \sin(x+\pi/2)$ and $0 < x < \pi$ (x is measured in radians), then $x =$ ________.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{3\pi}{4}$

(c) $\dfrac{2\pi}{3}$ (d) $\dfrac{\pi}{3}$

(e) None of these

Solution. (b) We first note that $\sin(x+\pi) = -\sin(x)$ and $\sin(x+\pi/2) = \cos(x)$. So, we are to find the solution to $-\sin(x) = \cos(x)$, which for $x \neq \pi/2$, is equivalent to $\tan(x) = -1$. The solution in the stated domain is $x = 3\pi/4$.

8. Suppose N is the smallest integer larger than 1 such that when divided by *every* $k = 2, 3, \ldots, 10$, the resulting remainder is 1. Then, ...

(a) $500 < N < 1000$ (b) $1000 < N < 1500$

(c) $1500 < N < 2000$ (d) $2000 < N < 2500$

(e) None of these

Solution. (e) So, $N - 1$ is divisible by every $k = 2, 3, \ldots, 10$, which means $N-1$ is divisible by the least common multiple of $k = 2, 3, \ldots, 10$, which is

$$2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520.$$

Thus, $N = 2521$, which is not in any of provided intervals.

9. Define a function f on *positive integers* by

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ 3x+1 & \text{if } x \text{ is odd.} \end{cases}$$

How many (integer) solutions are there to the equation

$$f(x) + f(x+1) = 2017?$$

(a) 0

(b) 1

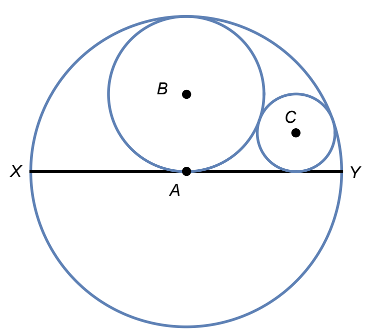
(c) 2

(d) 3

(e) None of these

Solution. **(a)** If x is even, then $f(x)+f(x+1) = x/2+3(x+1)+1 = 7x/2 + 4$. Solving $7x/2 + 4 = 2017 \implies x = 4026/7$, which is not an integer. If, on the other hand, x is odd, then $f(x) + f(x+1) = 3x+1+(x+1)/2 = 7x/2+3/2$. Again, solving $7x/2+3/2 = 2017 \implies x = 4031/7$, which is also not an integer. So, there are no (integer) solutions.

10. Let's label the three circles pictured below by their respective centers A, B, and C. Circle B is tangent to circle A and goes through the center point A and is tangent to the diameter $\overline{XY}$ of circle A. Circle C is mutually tangent to circles A and B and the diameter $\overline{XY}$. If the radius of circle A is R, then the radius of circle C is ________.

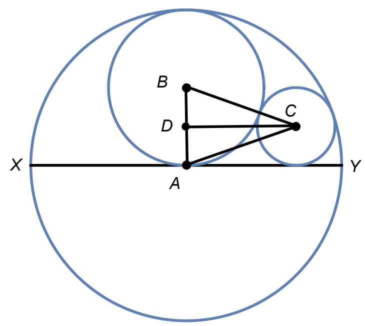


(a) $\dfrac{R}{2\sqrt{2}}$ (b) $\dfrac{R}{4}$

(c) $\dfrac{R}{2+\sqrt{2}}$ (d) $\dfrac{R}{4\sqrt{2}}$

(e) None of these

Solution. **(b)** Find point D on circle B radius $\overline{AB}$ so that $\overline{CD}$ is parallel to $\overline{XY}$.



Let r be the radius of circle C. Then, $AB = R/2$, $AD = r$ (so $BD = R/2 - r$), $AC = R - r$, and $BC = R/2 + r$. Using the Pythagorean Theorem on triangles ACD and BCD then gives us

$$(R/2 + r)^2 - (R/2 - r)^2 = CD^2 = (R - r)^2 - r^2.$$

Solving this equation gives

$$r = \frac{R}{4}.$$

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Part II Solutions

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11. The positive real number solution to the equation

$$\frac{x}{2017} - \frac{2017}{x} = 1$$

is ...

(a) $x = 2017(\sqrt{5}+1)$

(b) $x = 2017(\sqrt{5}-1)$

(c) $x = \dfrac{\sqrt{5}-1}{2017}$

(d) $x = \dfrac{\sqrt{5}+1}{2017}$

(e) None of these

Solution. (e)

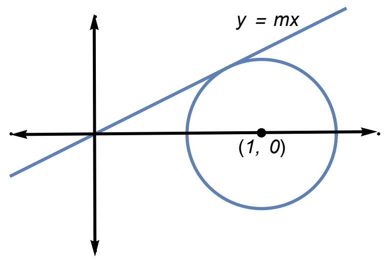
$$\begin{aligned} \frac{x}{2017} - \frac{2017}{x} = 1 &\implies x^2 - 2017x - 2017^2 = 0 \\ &\implies x = \frac{2017 \pm \sqrt{2017^2 - 4(-2017^2)}}{2} \\ &\implies x = \frac{2017(1 \pm \sqrt{5})}{2}. \end{aligned}$$

The positive solution is

$$x = \frac{2017(1+\sqrt{5})}{2},$$

none of the answer choices provided.

12. In the figure below, the circle centered at the point $(1, 0)$ is tangent to the line $y = mx$, where $m > 0$. Then, the radius of the circle is ________.



(a) $\dfrac{1}{\sqrt{m^2+1}}$

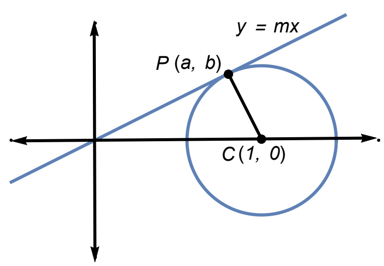
(b) $\dfrac{m+1}{\sqrt{m^2+1}}$

(c) $\dfrac{m^2}{\sqrt{m^2+1}}$

(d) $\dfrac{m}{\sqrt{m^2+1}}$

(e) None of these

Solution. **(d)** Label the circle center as C and let the point P of tangency of the circle and line have coordinates (a, b) as pictured below.



The radius $\overline{PC}$ is perpendicular to the tangent line, and so it has equation $y = (-1/m)(x - 1)$. The value of a is then the x solution to the equation $mx = (-1/m)(x - 1)$,

$$a = \frac{1}{m^2+1}.$$

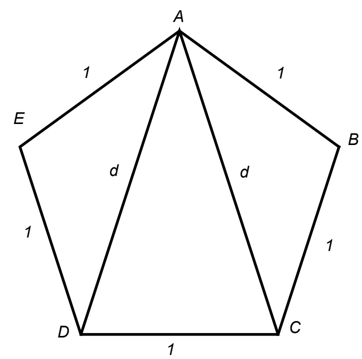
And, the value of b is ma,

$$b = \frac{m}{m^2+1}.$$

So, using the distance formula, the radius is

$$\begin{aligned} PC &= \sqrt{\left(\frac{1}{m^2+1} - 1\right)^2 + \left(\frac{m}{m^2+1}\right)^2} \\ &= \frac{m}{\sqrt{m^2+1}}. \end{aligned}$$

13. The pentagon $ABCDE$ pictured below is a *regular* pentagon with all five side lengths equal to 1. Let $d = AC = AD$. Then, $d =$ ______.



(a) $\dfrac{\sin 108°}{\sin 36°}$
(b) $\dfrac{2\sin 108°}{\sin 36°}$
(c) $\dfrac{\sin 108°}{2\sin 36°}$
(d) $\dfrac{2\sin 108°}{3\sin 36°}$
(e) None of these

Solution. **(a)** Since the pentagon is regular, $m\angle B = 108°$. Consequently, $m\angle BAC = m\angle BCA = 36°$. By the Law of Sines, $d/(\sin\angle B) = 1/(\sin\angle BAC)$. Thus,

$$d = \frac{\sin\angle B}{\sin\angle BAC} = \frac{\sin 108°}{\sin 36°}$$

14. How many multiples of 2017 with the units digit equal to 1 are there between 0 and 20,172,017?

(a) 999

(b) 1000

(c) 1001

(d) 1002

(e) None of these

Solution. **(b)** The units digit of $2017k$ is equal to 1 if, and only if, the units digit of $7k$ is equal to 1. And this happens when the units digit of k is 3. So, $k = 10m + 3$ for some non-negative integer m and since $20172017 = 10001 \cdot 2017$, k must be between 0 and 10001. Under this restriction, the possible values for m are $m = 0, 1, 2, \ldots 999$. Thus, there are 1000 multiples of 2017 with the units digit equal to 1 between 0 and 20,172,017

15. The solution to the inequality $-1 \leq |x-2| - |x-4| \leq 1$ is in the form $a \leq x \leq b$. Then, $a + b =$ ________.

(a) 4

(b) 5

(c) 6

(d) 7

(e) None of these

Solution. **(c)** We can rewrite the absolute value differenec as

$$|x-2| - |x-4| = \begin{cases} -2 & \text{if } x \leq 2 \\ 2x-6, & \text{if } 2 \leq x \leq 4, \\ 2 & \text{if } 4 \leq x. \end{cases}$$

Now we can see that a and b are the respective solutions to the equations $2x-6 = -1$ and $2x-6 = 1$. Thus, $a = 5/2$ and $b = 7/2$, giving $a+b = 6$.

16. A one percent increase in the diagonal length of a square results in what percentage increase in its area?

(a) 1.99%

(b) 2%

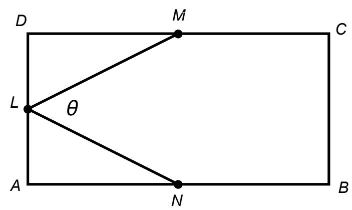
(c) 2.01%

(d) 2.02%

(e) None of these

Solution. **(c)** The area of a square with diagonal length d is $d^2/2$. If we increase the diagonal length by 1% we get the square area to be $(1.01d)^2/2 = (1.01)^2 d^2/2$. Now, $1.01^2 = 1.0201$ and so the increase in square area is 2.01%.

17. In the rectangle $ABCD$ pictured below, $AB = DC = a$, $AD = BC = b$, and L, M, N are the respective midpoints of $\overline{AD}, \overline{DC}, \overline{AB}$. Let $\theta = \text{m}\angle MLN$. Then, $\cos\theta =$ ________.



(a) $\dfrac{a-b}{a+b}$ (b) $\dfrac{a}{b}$

(c) $\dfrac{a^2-b^2}{a^2+b^2}$ (d) $\dfrac{a^2+b^2}{a^2-b^2}$

(e) None of these

Solution. **(c)** By the Pythagorean Theorem, we get

$$(LM)^2 = (LN)^2 = \frac{1}{4}(a^2+b^2).$$

And clearly, $MN = b$, so by the Law of Cosines applied to $\triangle MLN$, $(LM)^2 + (LN)^2 - 2(LM)(LN)\cos\theta = (MN)^2 \Longrightarrow$

$$\frac{1}{4}(a^2+b^2) + \frac{1}{4}(a^2+b^2) - 2\cdot\frac{1}{4}(a^2+b^2)\cos\theta = b^2 \quad \Longrightarrow$$

$$\frac{1}{2}(a^2+b^2) - \frac{1}{2}(a^2+b^2)\cos\theta = b^2 \quad \Longrightarrow$$

$$\cos\theta = \frac{a^2-b^2}{a^2+b^2}.$$

18. Lenny has \$5.85 in nickels, dimes and quarters in his pocket. Assuming he has 52 coins, what is the least number of nickels he could have?

(a) 1

(b) 2

(c) 3

(d) 4

(e) None of these

Solution. **(b)** Let the number of nickels, dimes, and quarters be represented by n, d and q, respectively. We then have two equations,

$$\begin{aligned} 5n + 10d + 25q &= 585 \\ n + d + q &= 52. \end{aligned}$$

Solving for d and q gives

$$\begin{aligned} d &= \frac{143 - 4n}{3} \\ q &= \frac{13 + n}{3}. \end{aligned}$$

The smallest value of n that yields positive integer values for d and q is $n = 2$, giving $d = 45$ and $q = 5$.

19. If you divide 2017 by 20, there results the remainder 17. Find the number of integers m larger than 17 (and smaller than 2017) for which if you divide 2017 by m, there results the remainder 17.

(a) 11

(b) 12

(c) 13

(d) 14

(e) None of these

Solution. **(c)** So we are looking for the integers m between 17 and 2017 for which $2017 = mq + 17$ for some other integer q. This, $2000 = mq$, which is to say, m is a divisor of 2000. Since $2000 = 2^4 \cdot 5^3$, all of the divisors of 2000 are in the form $2^a \cdot 5^b$ where $a = 0, 1, 2, 3, 4$ and $b = 0, 1, 2, 3$. Thus, there are 20 divisors of 2000. The divisors that are less than 17 are $1, 2, 4, 5, 8, 10, 16$ (seven divisors). Thus, there are 13 divisors larger than 17. That is, there are 13 possible values for m.

20. Suppose a, b, c, d are positive real numbers. Then,

$$\log_{\left(a^b\right)}\left(c^d\right) = \underline{\qquad\qquad}.$$

(a) $\dfrac{d\log_a c}{b}$

(b) $\dfrac{d\log_a c}{\log_a b}$

(c) $\dfrac{d\log_a c}{\log_b a}$

(d) $\dfrac{d\log_b c}{\log_a b}$

(e) None of these

Solution. (a)

$$\begin{aligned}
\log_{\left(a^b\right)}\left(c^d\right) &= d\log_{\left(a^b\right)} c \\
&= d\log_{\left(a^b\right)} a^{\log_a c} \\
&= d(\log_a c)\log_{\left(a^b\right)} a \\
&= d(\log_a c)\cdot\frac{1}{b} \\
&= \frac{d\log_a c}{b}.
\end{aligned}$$