# 2012
# Leap Frog Relay Grades 9-12
# Part I Solutions

**No calculators allowed**
**Correct Answer = 4, Incorrect Answer = $-1$, Blank = 0**

1. The Sale at Clothesmart is half-off, with an additional 10% off of the sale price. This is equivalent to what percentage off the original price?

   (a) 55% (b) 45%

   (c) 50% (d) 40%

   (e) None of these

   **Solution. (a)** Half off is a 50% sale. An additional 10% off the sale price would then be equivalent to a percentage reduction of $50+.1(50) = 50+5 = 55$.

2. Mrs. McGregor's garden has 10 times as many roses as azaleas and 5 times as many tulips as roses. After picking 16 roses, she finds that there are now 6 times as many tulips as roses and azaleas combined. How many tulips are there in Mrs. McGregor's garden?

   (a) 50 (b) 100

   (c) 150 (d) 250

   (e) None of these

**Solution.** **(e)** Let $R$, $A$ and $T$ represent the initial respective number of roses, azaleas and tulips. The given facts then translate to 3 equations in 3 unknowns...

$$\begin{aligned} R &= 10A \\ T &= 5R \\ T &= 6(A + R - 16). \end{aligned}$$

Solve this equation to obtain $R = 60$, $A = 6$ and $T = 300$, none of the answer choices provided.

3. What is the area of the triangle whose three side lengths are 2 inches, 3 inches and 4 inches respectively?

(a) Area $= \dfrac{3\sqrt{15}}{2}$ in$^2$

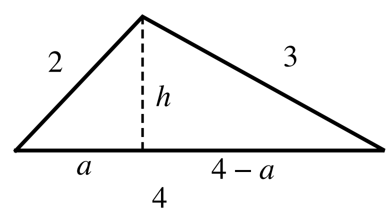
(b) Area $= \dfrac{3\sqrt{15}}{3}$ in$^2$

(c) Area $= \dfrac{3\sqrt{15}}{5}$ in$^2$

(d) Area $= \dfrac{3\sqrt{15}}{4}$ in$^2$

(e) None of these

**Solution.** **(d)** Label the figure as pictured, where $h$ is the height of the triangle.




By the Pythagorean Theorem, we have two equations

$$a^2 + h^2 = 4 \quad \text{and} \quad (4 - a)^2 + h^2 = 9.$$

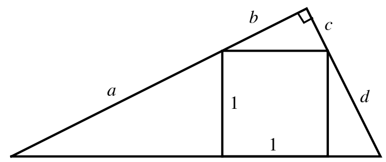
Solving, we get

$$a = \frac{11}{8} \quad \text{and} \quad h = \frac{3\sqrt{15}}{8}.$$

So, the area of the triangle is ($\frac{1}{2} \cdot \text{base} \cdot \text{height}$)

$$\frac{1}{2} \cdot 4 \cdot \frac{3\sqrt{15}}{8} = \frac{3\sqrt{15}}{4}.$$

4. The unit square is inscribed in the right triangle, with lengths $a, b, c, d$ as indicated.




Then, $ac + bd = \ldots$

(a) 3

(b) $\sqrt{2}$

(c) 2

(d) $2\sqrt{2}$

(e) None of these

**Solution.** **(c)** Using similar triangles, we get

$$\frac{a}{1} = \frac{1}{c} \Longrightarrow ac = 1 \quad \text{and} \quad \frac{d}{1} = \frac{1}{b} \Longrightarrow bd = 1.$$

This gives $ac = 1$ and $bd = 1$, which implies

$$ac + bd = 2.$$

5. $\cos(2012^\circ) = \ldots$

(a) $\cos^4(503^\circ) - \cos^2(503^\circ) + 1$

(b) $2\cos^4(503^\circ) + 2\cos^2(503^\circ)) + 1$

(c) $4\cos^4(503^\circ) - 4\cos^2(503^\circ) + 1$

(d) $8\cos^4(503^\circ) - 8\cos^2(503^\circ) + 1$

(e) None of these

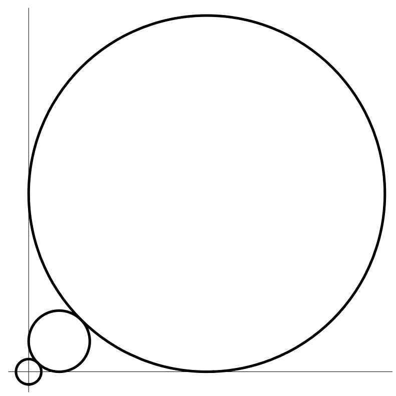
**Solution.** **(d)** Using double angle formulas, we have ...

$$\begin{aligned}
\cos(4\theta) &= 2\cos^2(2\theta) - 1 \\
&= 2(2\cos^2(\theta) - 1)^2 - 1 \\
&= 2(4\cos^4(\theta) - 4\cos^2(\theta) + 1) - 1 \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1.
\end{aligned}$$

Letting $4\theta = 2012^\circ$, so $\theta = 503^\circ$, gives us

$$\cos(2012^\circ) = 8\cos^4(503^\circ) - 8\cos^2(503^\circ) + 1.$$

6. In the figure below, the small circle is centered at the origin and has radius equal to 1. The middle circle is tangent to the small circle and the two axes. The large circle is tangent to the middle circle and also tangent to the axes. What is the radius of the large circle?



(a) $3\sqrt{2}+8$

(b) $7\sqrt{2}+4$

(c) $6\sqrt{2}+5$

(d) $5\sqrt{2}+7$

(e) None of these

**Solution. (d)** Let $r$ be the radius of the middle circle. Then by the Pythagorean Theorem, $2r^2 = (r+1)^2$. Solving this equation gives $r = \sqrt{2}+1$. Now let $R$ be the radius of the large circle. Again by the Pythagorean Theorem, we have an equation $2R^2 = (R+2r+1)^2$. Upon substituting $r = \sqrt{2}+1$ and solving, we obtain $R = 5\sqrt{2}+7$.

7. $(1+\log_{2011} 2012)(1-\log_{2012} 2011) = \ldots$

(a) $1-(\log_{2011} 2012)(\log_{2012} 2011)$

(b) $\log_{2011} 2012 - \log_{2012} 2011$

(c) $(\log_{2011} 2012)(\log_{2012} 2011)$

(d) $\log_{2011} 2012 + \log_{2012} 2011$

(e) None of these

**Solution. (b)** Use the identity

$$\log_b a = \frac{1}{\log_a b}.$$

$$\begin{aligned}(1+\log_a b)(1-\log_b a) &= 1+\log_a b-\log_b a-(\log_a b)(\log_b a)\\ &= 1+\log_a b-\log_b a-1\\ &= \log_a b-\log_b a.\end{aligned}$$

Thus,

$$(1+\log_{2011} 2012)(1-\log_{2012} 2011)=\log_{2011} 2012-\log_{2012} 2011.$$

8. $\sqrt{13}+\sqrt{7}=\ldots$

(a) $\sqrt{8+3\sqrt{91}}$

(b) $\sqrt{20+2\sqrt{91}}$

(c) $\sqrt{91-20\sqrt{8}}$

(d) $6\sqrt{20}-20$

(e) None of these

**Solution. (b)**

$$\begin{aligned}(\sqrt{13}+\sqrt{7})^2 &= 13+2\sqrt{13}\cdot\sqrt{7}+7\\ &= 20+2\sqrt{91}.\end{aligned}$$

And so,

$$\sqrt{13}+\sqrt{7}=\sqrt{20+2\sqrt{91}}.$$

9. The decimal (base 10) 3-digit number $n=(xyz)_{10}$, where $x\neq 0$, can be written in base 7 as $(abc)_7$ and in base 5 as $(cba)_5$ for some trio of digits $a,b,c$ that are chosen from $\{0,1,2,3,4\}$, and $a$ and $c$ are non-zero. Then, the sum of the decimal digits of $n$ is $x+y+z=\ldots$

(a) 7

(b) 4

(c) 2

(d) 3

(e) None of these

**Solution. (d)** The equality $(abc)_7=(cba)_5$ means $49a+7b+c=25c+5b+a$. Rearrange and divide by 2 to get

$$24a+b=12c.$$

It follows that $b$ is divisible by 12, and so $b = 0$. Substituting $b = 0$ into the above equation and dividing by 12 gives us

$$2a = c.$$

The only possibilities for $a$ are then 1 or 2. If $a = 1$, then $n = 51$, a 2-digit number. If $a = 2$, then $n = 102$, with digit sum

$$x + y + z = 3.$$

10. The one-digit number 9 has the property that its square, 81, has digit sum equal to $8 + 1 = 9$. How many two-digit numbers $N$ have digit sums of $N$ and $N^2$ both equal to 9?

(a) 1 (b) 2

(c) 3 (d) 4

(e) None of these

**Solution. (b)** The two-digit numbers whose digit sums are all equal to 9 are precisely the multiples of 9: 18, 27, 36, 45, 54, 63, 72 and 81. Their respective squares are $18^2 = 324$, $27^2 = 729$, $36^2 = 1296$, $45^2 = 2025$, $54^2 = 2916$, $63^2 = 3969$, $72^2 = 5184$ and $81^2 = 6561$. The only numbers that satisfy the condition are $N = 18$ and $N = 45$.

# 2012
# Leap Frog Relay Grades 9-12
# Part II Solutions

**No calculators allowed**
**Correct Answer = 4, Incorrect Answer = −1, Blank = 0**

1. Move forward 1 foot, then backwards 2 feet, then forward 3 feet, etc, ending with backwards 2012 feet. How may feet are you from your original starting point?

   (a) 1006 feet (b) 670 feet

   (c) 2012 feet (d) 503 feet

   (e) None of these

   **Solution. (a)** Each successive combination Forward/Backward places you 1 step behind. There are 1006 Forward/Backward combinations, so you end up 1006 feet from your original starting point.

2. How many solutions are there to the equation

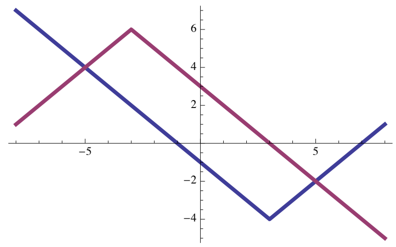
$$|x-3|-4=6-|x+3|?$$

   (a) 2 (b) 1

   (c) 0 (d) Infinitely many

   (e) None of these

**Solution. (a)** Graph the two curves $y = |x-3|-4$ and $y = 6-|x+3|$ to see there are 2 solutions $x = -5$ and $x = 5$.



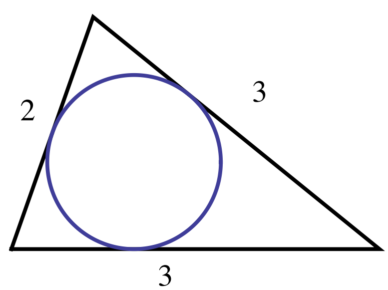

3. The solution to the equation $\log_3 x + \log_9 x + \log_{27} x = 2012$ is ...

   (a) $x = 3^{12070/11}$
   (b) $x = 3^{12072/11}$
   (c) $x = 3^{13072/11}$
   (d) $x = 3^{12082/11}$
   (e) None of these

**Solution. (b)** Use the change of base formula for logarithms...

$$\begin{aligned}
\log_3 x + \log_9 x + \log_{27} x = 2012 \quad &\Longrightarrow \quad \log_3 x + \frac{\log_3 x}{\log_3 9} + \frac{\log_3 x}{\log_3 27} = 2012 \\
&\Longrightarrow \quad (\log_3 x)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 2012 \\
&\Longrightarrow \quad (\log_3 x) = 2012 \cdot \frac{6}{11} \\
&\Longrightarrow \quad x = 3^{12072/11}.
\end{aligned}$$

4. The circle is inscribed in the triangle whose respective side lengths are 2, 3 and 3. Determine the radius of the circle.



(a) radius $= 1$

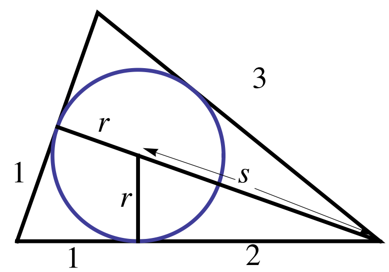
(b) radius $= \dfrac{2}{3}$

(c) radius $= \dfrac{1}{\sqrt{2}}$

(d) radius $= \dfrac{\sqrt{3}}{2}$

(e) None of these

**Solution.** **(c)** Label the radius $r$ and the length $s$ as pictured.




We then get, by the Pythagorean Theorem,

$$r^2 + 2^2 = s^2 \quad \text{and} \quad (r+s)^2 + 1^2 = 3^2.$$

The second equation implies $r + s = \sqrt{8}$. Substituting $s = \sqrt{8} - r$ into the first equation then yields

$$r = \frac{1}{\sqrt{2}}.$$

5. The sum of the divisors of 2012 is ...

(a) 3524

(b) 3526

(c) 3528

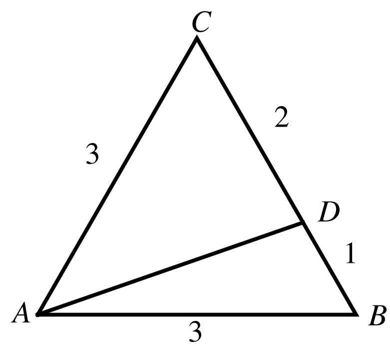
(d) 3530

(e) None of these

**Solution.** **(c)** The factorization of 2012 as power of primes is $2012 = 2^2 \cdot 503$. (Note 503 is prime because it is not divisible by any of the primes less that $\sqrt{503}$, which are 2, 3, 5, 7, 11, 13, 17, 19.) The divisor sum of 2012 is then

$$1 + 2 + 2^2 + 503 + 2 \cdot 503 + 2^2 \cdot 503 = 3528.$$

6. In the figure below, $\triangle ABC$ is an equilateral triangle with side lengths all equal to 3. If $BD = 1$, the $AD = \ldots$




(a) $\frac{5}{2}$

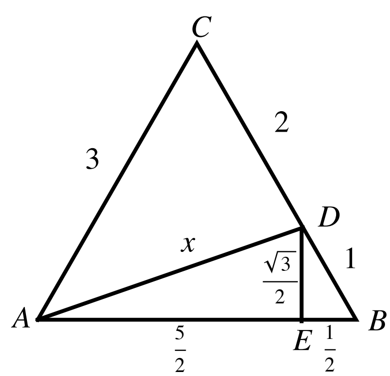
(b) $2\sqrt{2}$

(c) $\sqrt{7}$

(d) $1 + \sqrt{3}$

(e) None of these

**Solution.** **(c)** Drop a perpendicular $\overline{DE}$ as pictured.




Because $\triangle EBD$ is a $30° - 60° - 90°$ triangle, we have $DE = \sqrt{3}/2$ and $EB = 1/2$. And so $AE = 5/2$, so we can use the Pythagorean Theorem to determine $x = AD$,

$$x^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 7.$$

We obtain $AD = x = \sqrt{7}$.

7. The rational number $q$ has the (infinite repeating) base-3 representation $(0.101010\ldots)_3$. What is the base-2 representation of $q$?

(a) $(0.110110110\ldots)_2$

(b) $(0.010101\ldots)_2$

(c) $(0.111)_2$

(d) $(0.011)_2$

(e) None of these

**Solution.** **(d)** We can solve for $q$ using the geometric series ...

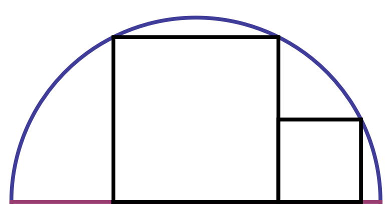
$$\begin{aligned} q &= \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \cdots \\ &= \frac{1}{3}\left(1 + \frac{1}{9} + \frac{1}{9^2} + \cdots\right) \\ &= \frac{1}{3}\left(\frac{1}{1-\frac{1}{9}}\right) \\ &= \frac{3}{8}. \end{aligned}$$

Thus,

$$q = \frac{3}{8} = \frac{1}{4} + \frac{1}{8},$$

with base-2 (finite) representation $(0.011)_2$.

8. In the figure below, the semicircle has radius equal to 1 inch, and the two adjacent squares are inscribed as pictured. What is the area of the smaller square?



(a) Area $= \dfrac{1}{4}$ in$^2$

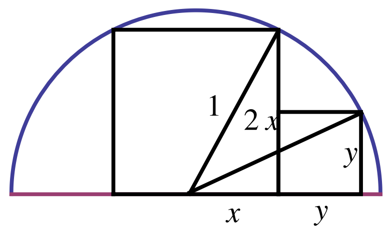
(b) Area $= \dfrac{1}{\sqrt{5}}$ in$^2$

(c) Area $= \dfrac{2}{\sqrt{7}}$ in$^2$

(d) Area $= \dfrac{1}{1+\sqrt{5}}$ in$^2$

(e) None of these

**Solution.** **(e)** Label the figure as pictured below.




By the Pythagorean Theorem, we have the two equations

$$x^2 + (2x)^2 = 1 \quad \text{and} \quad (x+y)^2 + y^2 = 1.$$

The first equation gives us $x = 1/\sqrt{5}$. Substituting this into the second equation gives,

$$2y^2 + \frac{2}{\sqrt{5}}y - \frac{4}{5} = 0.$$

Use the quadratic formula, and taking the positive solution, to get

$$y = \frac{1}{\sqrt{5}}.$$

Thus, the area of the smaller square is $y^2 = 1/5$ in$^2$, none of the the answer choices provided.

9. Happy Cone Ice Cream Parlor sells 24 flavors of ice cream. A group of $N$ students come in to buy ice cream. Assuming each student buys one flavor, what is the *smallest* value of $N$ that will *guarantee* at least 4 of the students will buy the same flavor?

(a) 24 (b) 48

(c) 72 (d) 96

(e) None of these

**Solution.** **(e)** The smallest value of $N$ to guarantee 4 students receiving the same flavor is 73, none of the answer choices provided. First of all, $N = 72$ will not suffice, as it would then be possible for each flavor to be chosen by only 3 students, and $3 \times 24 = 72$. To see that $N = 73$

will suffice, *assume* to the contrary that $N = 73$ *and* no 4 students choose the same flavor. Then there would be at most $3 \times 24 = 72$ students, contradicting our assumption.

10.
$$\frac{\cos(2012^\circ) + \sin(2012^\circ)}{\cos(2012^\circ) - \sin(2012^\circ)} = \dots$$

(a) $\csc(64^\circ) - \cot(64^\circ)$

(b) $\csc(64^\circ) + \cot(64^\circ)$

(c) $\sec(64^\circ) - \tan(64^\circ)$

(d) $\sec(64^\circ) + \tan(64^\circ)$

(e) None of these

**Solution. (d)**

$$\begin{aligned}
\frac{\cos(x^\circ) + \sin(x^\circ)}{\cos(x^\circ) - \sin(x^\circ)} &= \frac{\cos(x^\circ) + \sin(x^\circ)}{\cos(x^\circ) - \sin(x^\circ)} \cdot \frac{\cos(x^\circ) + \sin(x^\circ)}{\cos(x^\circ) + \sin(x^\circ)} \\
&= \frac{\cos^2(x^\circ) + 2\cos(x^\circ)\sin(x^\circ) + \sin^2(x^\circ)}{\cos^2(x^\circ) - \sin^2(x^\circ)} \\
&= \frac{1 + \sin(2x^\circ)}{\cos(2x^\circ)} \\
&= \sec(2x^\circ) + \tan(2x^\circ).
\end{aligned}$$

If $x = 2012$, then $2x = 4024 = 11 \times 360 + 64$. And so,

$$\frac{\cos(x^\circ) + \sin(x^\circ)}{\cos(x^\circ) - \sin(x^\circ)} = \sec(64^\circ) + \tan(64^\circ).$$