2011
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. The iPop and iCase cost \$100 together. If the iPop costs \$90 more than the iCase, then the iCase costs...

   (a) \$15 (b) \$20

   (c) \$10 (d) \$5

   (e) None of these

   Solution. (d) If the respective costs of the iPoP and iCase are p and c, then we have two equations: $c + p = 100$ and $p = 90 + c$. We solve for $c = 5$. So the iCase costs \$5.

2. How many real number solutions are there to the equation

$$|2010x + 2011| = |2011x + 2010|?$$

   (a) 0 (b) 1

   (c) 2 (d) 4

   (e) None of these

Solution. (c) There are two cases, $2010x + 2011 = 2011x + 2010$ and $2010x + 2011 = -2011x - 2010$. Each case leads to a single solution, giving a total of 2 solutions.

3. Lenny melts 2011 $1''$ by $1''$ by $1''$ ice cubes and refreezes the water to form one large ice cube (all side lengths equal). The side length of the large cube is

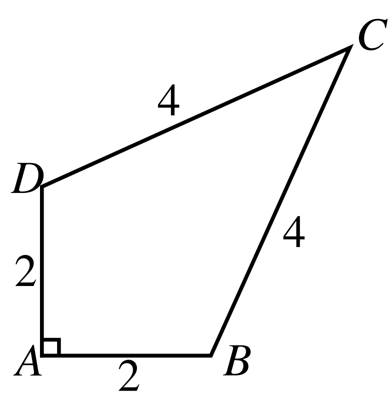
(a) between 10 and 11 inches. (b) between 11 and 12 inches.

(c) between 12 and 13 inches. (d) between 13 and 14 inches.

(e) None of these

Solution. (c) The volume of the combined water from the cubes is 2011 cubic inches. Since $12^3 = 1728$ and $13^3 = 2197$ we can see that the side length of the large cube is between 12 and 13 inches.

4. In the figure below, the lengths are as labeled and the angle at A is a right angle. The area enclosed by $ABCD$ is ...



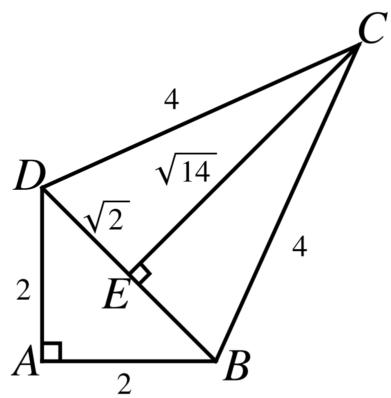
(a) $4 + 2\sqrt{7}$ (b) $2 + 2\sqrt{6}$

(c) $2 + 2\sqrt{7}$ (d) $4 + \sqrt{6}$

(e) None of these

Solution. (c) Draw the segment $\overline{BD}$ and drop a perpendicular from

C to a point E on $\overline{BD}$. Then E is the midpoint of $\overline{BD}$ and by the Pythagorean Theorem, we get $BD = 2\sqrt{2}$ and $CE = \sqrt{14}$.



We compute the area as the sum of the two triangle areas $\triangle ABD$ and $\triangle BDC$,

$$\frac{1}{2}2 \cdot 2 + \frac{1}{2}2\sqrt{2} \cdot \sqrt{14} = 2 + 2\sqrt{7}.$$

5. The solution, x, to the equation

$$\log_2 x + \log_4 x = 2010$$

is in the form $x = 2^N$ where ...

(a) $1 \leq N < 300$

(b) $300 \leq N < 600$

(c) $600 \leq N < 900$

(d) $900 \leq N < 1200$

(e) None of these

Solution. (e) Convert the equation to base 2,

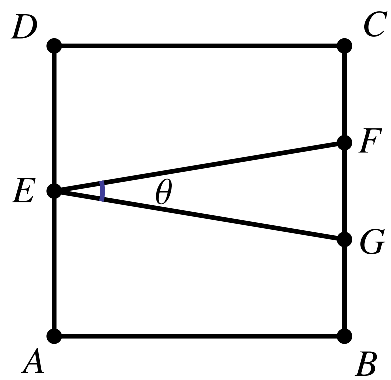
$$\log_2 x + \frac{\log_2 x}{\log_2 4} = 2010,$$

and use the fact $\log_2 4 = 2$ to then solve for $\log_2 x$,

$$\log_2 x = 1340.$$

So $x = 2^{1340}$, none of the answer choices presented.

6. In the square $(ABCD)$ pictured below, the point E is the midpoint of segment $\overline{AD}$ and the points F and G divide the segment $\overline{BC}$ into 3 equal length pieces. Then $\theta = \ldots$



(a) $\arctan\left(\frac{1}{6}\right)$

(b) $2\arctan\left(\frac{1}{3}\right)$

(c) $2\arctan\left(\frac{1}{6}\right)$

(d) $\arctan\left(\frac{1}{3}\right)$

(e) None of these

Solution. (c) If we let M be the midpoint of the segment $\overline{FG}$, then $\tan\angle FEM = \tan(\theta/2) = 1/6$. So,

$$\theta = 2\arctan\left(\frac{1}{6}\right).$$

7. The real numbers $2-\sqrt{3}$ and $5-\sqrt{7}$ are the roots of a fourth degree polynomial with integer coefficients $p(x) = x^4 + bx^3 + cx^2 + dx + e$. Then $e = \ldots$.

(a) 15

(b) 18

(c) 14

(d) 21

(e) None of these

Solution. (b) The roots to a polynomial with integer coefficients come in conjugate pairs and since the coefficient to x^4 is 1, e is the product of the roots. So the roots are $2 \pm \sqrt{3}$, $5 \pm \sqrt{7}$, whose product is

$$\begin{aligned}(2-\sqrt{3})(2+\sqrt{3})(5-\sqrt{7})(5+\sqrt{7}) &= (4-3)(25-7)\\ &= 18.\end{aligned}$$

8. How many distinct (no two the same) prime factors are there of the number that is the least common multiple of 1776 and 2011?

(a) 3
(b) 4
(c) 5
(d) 6
(e) None of these

Solution. (b) We can factor $1776 = 2^4 \cdot 3 \cdot 37$. Also, 2011 is prime. This can be seen by checking divisibility by all the primes less than $\sqrt{2011}$: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43. Thus, the least common multiple of 1776 and 2011 is their product, with prime factorization:

$$1776 \cdot 2011 = 2^4 \cdot 3 \cdot 37 \cdot 2011.$$

9. If the vertex of the parabola $y = x^2 + 2x + c$ lies on the line $y = 2010x + 2011$, then $c = \ldots$.

(a) 0
(b) 1
(c) 2011
(d) -2010
(e) None of these

Solution. (e) The vertex of the parabola $y = x^2 + 2x + c$ can be determined by completing the square,

$$\text{Vertex} = (-1, c-1)\,.$$

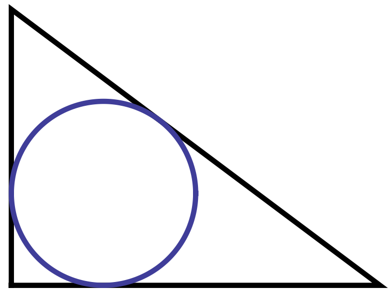
Substitute this into the equation of the given line,

$$c - 1 = 2010(-1) + 2011,$$

and solve for c,

$$c = 2.$$

10. In the figure below, the triangle is a right triangle with respective leg lengths a and b and hypotenuse length c. The diameter (d) of the inscribed circle is then...



(a) $d = \dfrac{a+b+c}{2}$

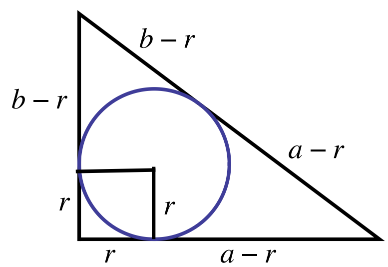
(b) $d = \sqrt{a^2+b^2+c^2}$

(c) $d = a+b-c$

(d) $d = (a+b)^2 - c^2$

(e) None of these

Solution. **(c)** From the figure below, we can see that $c = (a-r) + (b-r)$.



Solve for $d = 2r$ to get $d = a+b-c$.

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Part II Solutions

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1. Suppose the successive discounts of 10% and x% is equivalent to the single discount of 19%? Then x satisfies...

   (a) $9 < x \leq 10$ (b) $10 < x \leq 11$

   (c) $11 < x \leq 12$ (d) $12 < x \leq 13$

   (e) None of these

   Solution. (a) The variable x must satisfy the equation

   $$.9(1 - \frac{x}{100}) = .81.$$

   The solution is $x = 10$.

2. Suppose the graph of a parabola has vertex equal to $(1, 0)$ and goes through the point $(3, 3)$. Then, then y-intercept of the parabola is...

   (a) $\left(0, \frac{1}{4}\right)$ (b) $\left(0, \frac{3}{4}\right)$

   (c) $\left(0, \frac{1}{3}\right)$ (d) $\left(0, \frac{2}{3}\right)$

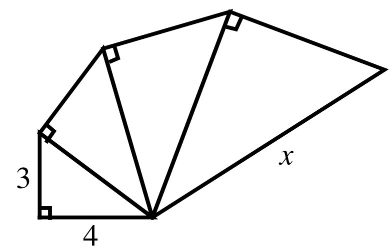
   (e) None of these

Solution. **(b)** Since the vertex is $(1,0)$, we can write the equation in the form $y = a(x-1)^2$. The graph goes through the point $(3,3)$, and so $3 = a(3-1)^2$. Solve for $a = 3/4$, to get the equation

$$y = \frac{3}{4}(x-1)^2.$$

Set $x = 0$ to obtain the y-intercept $\left(0, \frac{3}{4}\right)$.

3. In the figure below, all 4 triangles are right triangles (as indicated) and mutually similar. Determine the unknown length x.

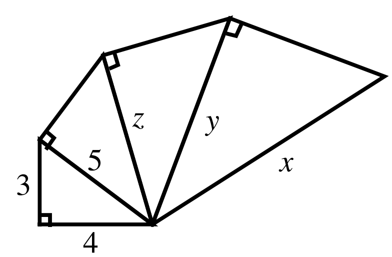


(a) $\frac{125}{8}$ (b) $\frac{625}{16}$

(c) $\frac{125}{16}$ (d) $\frac{625}{32}$

(e) None of these

Solution. **(e)** Label the figure...



Then, by similar triangles, we have

$$\frac{z}{5} = \frac{5}{4} \implies z = \frac{25}{4}$$

$$\frac{y}{z} = \frac{5}{4} \implies y = z \cdot \frac{5}{4} = \frac{125}{16}$$

$$\frac{x}{y} = \frac{5}{4} \implies x = y \cdot \frac{5}{4} = \frac{625}{64}.$$

4. The non-zero solution, x, to the equation

$$2^{x^3} = 3^{x^2}$$

satisfies...

(a) $0 < x \leq 1$

(b) $1 < x \leq 2$

(c) $2 < x \leq 3$

(d) $3 < x \leq 4$

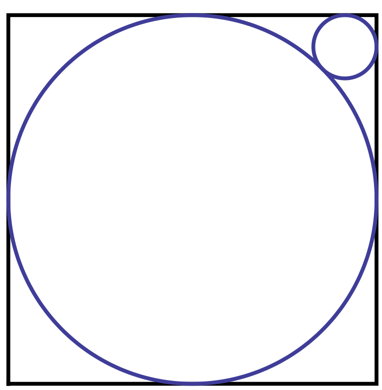
(e) None of these

Solution. (b) Convert the right-hand side to base 2,

$$\begin{aligned} 2^{x^3} = 3^{x^2} &\iff 2^{x^3} = 2^{\log_2 3^{x^2}} \\ &\iff 2^{x^3} = 2^{x^2 \log_2 3} \\ &\iff x^3 = x^2 \log_2 3. \end{aligned}$$

If $x \neq 0$, then we get the solution $x = \log_2 3$, which satisfies $1 < x \leq 2$.

5. In the figure below, the two circles are so inscribed in the square. If the dimensions of the square are 2 inches by 2 inches then the radius of the smaller circle is how many inches?



(a) $3-2\sqrt{2}$ inches

(b) $6-3\sqrt{2}$ inches

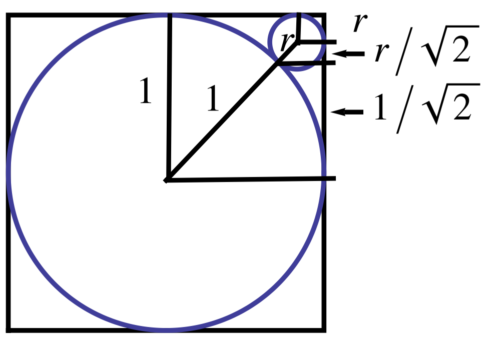
(c) $\sqrt{3}$ inches

(d) $\sqrt{5}$ inches

(e) None of these

Solution. **(a)** From the labeled picture below, we can see

$$r+\frac{r}{\sqrt{2}}+\frac{1}{\sqrt{2}}=1.$$



Solving for r, we get

$$r=3-2\sqrt{2}.$$

6. If N is the number of (decimal) digits in the number 2011^{2011}, then...

(a) $0<N\leq 3000$

(b) $3000<N\leq 6000$

(c) $6000<N\leq 9000$

(d) $9000<N\leq 12000$

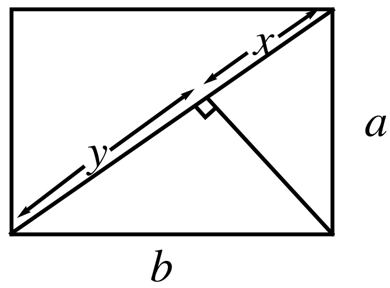
(e) None of these

Solution. **(c)** The number of digits of a number a is equal to the integer part of $1+\log_{10} a$. Thus, to determine the number of digits of 2011^{2011} we must get an estimate for $\log_{10} 2011^{2011}$. First note that

$$\log_{10} 2011^{2011} = 2011 \log_{10} 2011$$

and $\log_{10} 2011$ is a number between 3 and 4. So, $\log_{10} 2011^{2011}$ is a number between 6033 and 8044. So the number of digits of 2011^{2011} is between 6034 and 8045.

7. Determine the ratio x/y as a function of a and b from the figure (rectangle) below.



(a) $\dfrac{x}{y} = \dfrac{a^2}{b^2}$

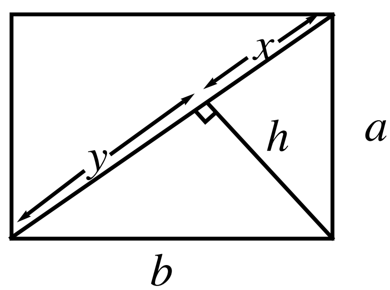
(b) $\dfrac{x}{y} = \dfrac{a}{b}$

(c) $\dfrac{x}{y} = \dfrac{a+b}{a^2+b^2}$

(d) $\dfrac{x}{y} = \dfrac{a+b}{\sqrt{a^2+b^2}}$

(e) None of these

Solution. (a) Using the labeled figure below, and noting that all three right triangles are mutually similar, we get ratios...



$$\frac{a}{b} = \frac{x}{h} \implies x = \frac{a}{b} \cdot h$$

and

$$\frac{b}{a} = \frac{y}{h} \implies y = \frac{b}{a} \cdot h$$

Putting this together gives

$$\frac{x}{y} = \frac{a^2}{b^2}.$$

8. Suppose a, b, c, d are real numbers that satisfy $a < b$ and $c < d$. Then it is necessary that...

(a) $ad + ac < bc + bd$

(b) $ac < bd$

(c) $ad + bc < bd + ac$

(d) $ab < cd$

(e) None of these

Solution. (c) Rewrite the two inequalities as $0 < b - a$ and $0 < d - c$. Then multiply and simplify

$$0 < (b-a)(d-c) \Longrightarrow ad + bc < bd + ac.$$

9. Suppose a, b, c, d are non-zero real numbers and $p(x) = ax + b$ and $q(x) = cx + d$. Suppose further the graphs of $y = p(q(x))$ and $y = q(p(x))$ coincide. Then a, b, c, d satisfy the identity...

(a) $ab + c = bc + d$

(b) $a + cd = b + cd$

(c) $bc + a = ab + d$

(d) $ad + b = bc + d$

(e) None of these

Solution. (d) Simplifying, we get $p(q(x)) = acx + ad + b$ and $q(p(x)) = acx + bc + d$. Setting these equal, we get

$$acx + ad + b = acx + bc + d \Longrightarrow ad + b = bc + d.$$

10. $\cos(15°) + \sin(15°) = \ldots$

(a) $1 + \sqrt{\frac{2}{3}}$

(b) $\sqrt{\frac{2}{3}}$

(c) $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

(d) $\sqrt{\frac{3}{2}}$

(e) None of these

Solution. **(d)** Let $y = \cos(15°) + \sin(15°)$. Then (using the identities $\cos^2\theta + \sin^2\theta = 1$ and $2\cos\theta\sin\theta = \sin(2\theta)$)

$$\begin{aligned}
y^2 &= (\cos(15°) + \sin(15°))^2 \\
&= \cos^2(15°) + 2\cos(15°)\sin(15°) + \sin^2(15°) \\
&= (\cos^2(15°) + \sin^2(15°)) + 2\cos(15°)\sin(15°) \\
&= 1 + \sin(30°) \\
&= 1 + \frac{1}{2} \\
&= \frac{3}{2}.
\end{aligned}$$

Finaly, since $y > 0$ we have

$$y = \sqrt{\frac{3}{2}}.$$