2008
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. $\log_9 \log_8 \log_5 25 =$ ______

   (a) $\frac{1}{3}$ (b) $-\frac{2}{3}$

   (c) $\frac{1}{2}$ (d) $-\frac{3}{2}$

   (e) None of these

   Solution. (e)

$$\begin{aligned} \log_9 \log_8 \log_5 25 &= \log_9 \log_8 2 \\ &= \log_9 \frac{1}{3} \\ &= -\frac{1}{2}, \end{aligned}$$

   none of the answer choices provided.

2. The base-2 number (repeated decimal) $.\overline{01}_2 = .010101\ldots_2$ is equal to ______.

   (a) $\frac{1}{3}$ (b) $\frac{1}{4}$

   (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

   (e) None of these

   Solution. (a)

$$.010101\ldots_2 = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \\
&= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) \\
&= \frac{1}{3}.
\end{aligned}
$$

3. Lenny has a jar of coins. The 1000 coins in the jar consists of nickels, dimes and quarters (at least one of each). Assuming that the value of the coins is exactly 100 dollars, what is the ratio of the number of nickels to the number of quarters?

   (a) 3 (b) 4

   (c) 5 (d) 6

   (e) None of these

   Solution. **(a)** Let N, D and Q be the respective numbers of nickels, dimes and quarters. There are 1000 coins whose total value is \$100, which gives the two equations,

$$
\begin{aligned}
N + D + Q &= 1000 \\
5N + 10D + 25Q &= 10000.
\end{aligned}
$$

   Divide the second equation by 5,

$$N + 2D + 5Q = 2000,$$

   and multiply the first equation by 2,

$$2N + 2D + 2Q = 2000.$$

   Thus the left hand sides of the above two equations are equal,

$$N + 2D + 5Q = 2N + 2D + 2Q.$$

   This simplifies to

$$3Q = N,$$

   so the ratio of nickels to quarters is 3.

4. Suppose $\sin(\theta + 90^\circ) = \cos(\theta + 45^\circ)$. Then $\tan\theta =$ ________.

   (a) $2 - \sqrt{2}$ (b) $\sqrt{2} - 2$

   (c) $1 - \sqrt{2}$ (d) $\sqrt{2} - 1$

   (e) None of these

Solution. **(c)** Using the angle addition formulas, we get

$$\begin{aligned} \sin(\theta + 90^\circ) &= \cos\theta \quad \text{and} \\ \cos(\theta + 45^\circ) &= \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta) \end{aligned}$$

So we have the equation

$$\cos\theta = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$$

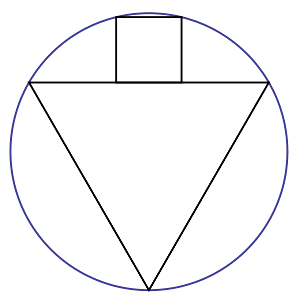
which can be rewritten as

$$(1 - \sqrt{2})\cos\theta = \sin\theta.$$

We know that $\cos\theta \neq 0$ for otherwise the above equation would imply $\sin\theta = 0 = \cos\theta$, which is impossible. So we may divide by $\cos\theta$ to obtain

$$\tan\theta = 1 - \sqrt{2}.$$

5. In the accompanying figure, the circle has radius equal to 1 unit, the inscribed triangle is equilateral and the inscribed rectangle is a square. The side length of the square is then ________ units.



(a) $\frac{1}{\sqrt{2}}$

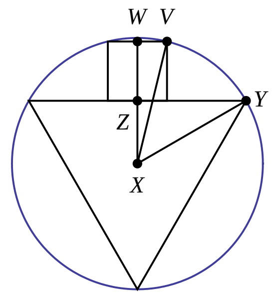
(b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{\sqrt{3}}$

(e) None of these

Solution. **(e)** Label the figure as pictured, where X denotes the center of the circle and the segment $\overline{WX}$ is perpendicular to the segment $\overline{WV}$.



By assumption, $XY = 1$. Since the triangle is equilateral, $\triangle XYZ$ is a 30–60–90 triangle, and consequently $XZ = 1/2$. Let s denote the side length of the square, $s = WZ$. Then $WX = 1/2 + s$. On the other hand, WX is one leg of the right triangle XVW, whose hypotenuse is $XV = 1$ and whose other leg is $WV = s/2$. So by the pythagorean theorem, $WX = \sqrt{1 - s^2/4}$. Setting the two values for WX equal, we obtain an equation that we can solve for s,

$$\frac{1}{2} + s = \sqrt{1 - \frac{s^2}{4}}.$$

Square both sides of the equation, rearrange terms and multiply by 4 to get

$$5s^2 + 4s - 3 = 0.$$

By the pythagorean theorem,

$$\begin{aligned} s &= \frac{-4 \pm \sqrt{4^2 - 4(5)(-3)}}{2(5)} \\ &= \frac{-2 \pm \sqrt{19}}{5}. \end{aligned}$$

Choose the + square root to obtain a positive answer,

$$s = \frac{-2 + \sqrt{19}}{5},$$

none of the answer choices provided.

6. $\sqrt{5 - 2\sqrt{6}} =$ ________.

(a) $4 - 2\sqrt{3}$

(b) $3 - 2\sqrt{2}$

(c) $2 - \sqrt{3}$

(d) $\sqrt{3} - \sqrt{2}$

(e) None of these

Solution. **(d)**

$$\begin{aligned} 5-2\sqrt{6} &= 3-2\sqrt{6}+2 \\ &= \left(\sqrt{3}-\sqrt{2}\right)^2. \end{aligned}$$

Take the positive square root to get

$$\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}.$$

7. The average of five 100 point tests is 96. What is the smallest possible average of four of the tests?

(a) 95

(b) 90

(c) 85

(d) 80

(e) None of these

Solution. **(a)** The smallest average of four will occur if the fifth is as large as possible, namely 100. Since the average of the five is 96, the sum of the points of the five tests will be $5 \times 96 = 480$. If one score is 100, then the remaining four scores will sum to $480 - 100 = 380$, whose average is $380/4 = 95$.

8. You flip a fair coin successively until it comes up Heads. What is the probability that it takes at most 3 flips to come up Heads?

(a) $\frac{5}{8}$

(b) $\frac{7}{8}$

(c) $\frac{3}{4}$

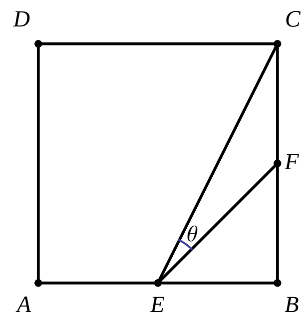
(d) $\frac{5}{6}$

(e) None of these

Solution. **(b)** The answer is one minus the probability that the first 3 flips are Tails. The probability that the first three flips are tails is $\frac{1}{8}$. Thus the probability of no more than 3 flips is

$$1-\frac{1}{8} = \frac{7}{8}.$$

9. Quadrilateral $ABCD$ is a square. The points E and F are the respective midpoints of $\overline{AB}$ and $\overline{BC}$. Suppose $\theta = \text{m}\angle FEC$. Determine $\tan\theta$.



(a) $\sqrt{\dfrac{2}{3}}$

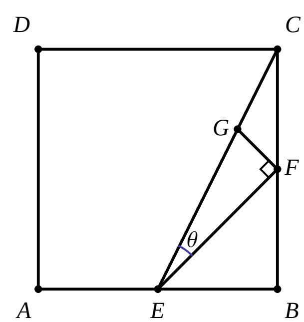
(b) $\dfrac{\sqrt{2}}{3}$

(c) $\dfrac{1}{3}$

(d) $\dfrac{1}{\sqrt{3}}$

(e) None of these

Solution. (c) Draw point G on $\overline{EC}$ so that $\overline{EF}$ is perpendicular to $\overline{GF}$.



Then $\tan\theta = GF/EF$. We may scale the square so that the side lengths are equal to 2 units. So $EB = BF = 1$, and we have, by the pythagorean theorem,

$$EF = \sqrt{2}.$$

So we are left with determining GF. Place the figure in a (x, y)–coordinate system so that E is the origin. Then $\overline{GF}$ has equation $y = -x + 2$ and $\overline{EC}$ has equation $y = 2x$. Solving these two equations gives the coordinates of $G = (2/3, 4/3)$. We can then use the distance formula to compute GF (where $F = (1, 1)$),

$$\begin{aligned} GF &= \sqrt{\left(\frac{2}{3} - 1\right)^2 + \left(\frac{4}{3} - 1\right)^2} \\ &= \frac{\sqrt{2}}{3}. \end{aligned}$$

Thus

$$\tan\theta = \frac{GF}{EF}$$

$$
\begin{aligned}
&= \frac{\sqrt{2}/3}{\sqrt{2}} \\
&= \frac{1}{3}.
\end{aligned}
$$

Alternative Solution. Let $\alpha = \angle BEC$ and $\beta = \angle BEF$. Then

$$
\begin{aligned}
\tan\theta &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\
&= \frac{2-1}{1+2\cdot 1} \\
&= \frac{1}{3}.
\end{aligned}
$$

10. Let a, b and c be the three roots of $x^3 + 2x + 3 = 0$. Then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = \underline{\qquad\qquad}.$$

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$

(c) $\frac{1}{3}$ (d) $\frac{1}{2}$

(e) None of these

Solution. **(a)** First note that

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - (abc),$$

Comparing this to $x^3 + 2x + 3$, we obtain the three equations

$$
\begin{aligned}
a+b+c &= 0 \\
ab+bc+ac &= 2 \\
abc &= -3.
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} &= -\left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right) \\
&= -\left(\frac{ab+bc+ac}{abc}\right)
\end{aligned}
$$

$$= -\left(\frac{2}{-3}\right)$$

$$= \frac{2}{3}.$$

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Part II Solutions

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11. The sum of the positive divisors of 2008 is ________. (Recall that among all divisors of a number N, we include 1 and N.)

(a) 3750 (b) 3760

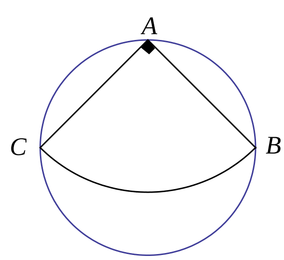
(c) 3770 (d) 3780

(e) None of these

Solution. (d) Using a factor tree, it is easy to see that $2008 = 2^3 \times 251$. It is also easy to see that 251 is prime by checking the only possible prime divisors all less than $\sqrt{251} < 16$. Thus, the divisors of 2008 are all of the form $2^a \times 251^b$ where $a = 0, 1, 2, 3$ and $b = 0, 1$,

$$1 + 2 + 4 + 8 + 251 + 2 \times 251 + 4 \times 251 + 8 \times 251 = 3780.$$

12. The right-angled circle sector (ABC) is inscribed in the circle as pictured below. Determine the ratio of the circle area to the sector area.

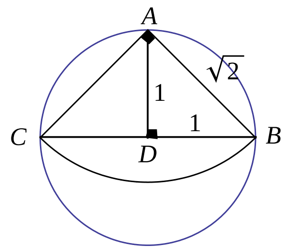


(a) $\sqrt{2}$ (b) 2

(c) $\dfrac{1}{\sqrt{2}}$ (d) $\dfrac{3}{\sqrt{2}}$

(e) None of these

Solution. **(b)** We may assume the radius of the circle is equal to 1 unit., so its area is equal to π square units. Since $m\angle A = 90°$, $\overline{BC}$ is a diameter of the circle. The radius of the circle sector is then $\sqrt{2}$, and so its area is $\frac{1}{4}\pi(\sqrt{2})^2 = \pi/2$ square units. The ratio is then $\pi/(\pi/2) = 2$.



13. Successive discounts of 10%, 20% and 30% is equivalent to a single discount of ________.

    (a) 49.9% (b) 49.8%

    (c) 49.7% (d) 49.6%

    (e) None of these

    Solution. **(d)** It is easiest if you look at the resulting discount on 100 dollars. The first discount (10%) results in a price of $100 - (.1)100 = 90$ (dollars). The second discount results in a price of $90 - (.2)90 = 72$ (dollars). The third discount results in a price of $72 - (.3)72 = 50.4$ (dollars). So the savings on 100 dollars is 49.6 dollars, which is a discount of 49.6%.

14. In exactly how many minutes after 12 noon do the minute and hour hands of a clock form a 180° angle for the first time?

    (a) $32\frac{8}{11}$ (b) $32\frac{9}{11}$

    (c) $32\frac{10}{11}$ (d) 33

    (e) None of these

    Solution. **(a)** The minute hand will travel $6m$ degrees in m minutes. The hour hand moves 12 times as slow, and so will travel through $6m/12 = m/2$ degrees in m minutes. The difference between the two is $6m - m/2 = (11/2)m$. Solve the equation $(11/2)m = 180$ to get $m = 360/11 = 32\frac{8}{11}$ minutes.

    Alternate Solution. The hour hand and minute hand will point in the same direction exactly 11 times during a 12 hour period in equally spaced intervals. So this will occur every 12/11 hours. The hands will be first at a 180° angle at the half way point, $\frac{1}{2} \cdot \frac{12}{11} = \frac{6}{11}$ of an hour. In minutes, this is

$$\begin{aligned} \frac{6}{11} \cdot 60 &= \frac{360}{11} \\ &= 32\frac{8}{11}. \end{aligned}$$

15. The number $4^6 + 1$ is the product of two primes. The larger of the two primes is a 3-digit number whose three digits sum to ______.

(a) 8
(b) 6
(c) 4
(d) 2
(e) None of these

Solution. (e) We can write $4^6 + 1 = (4^2)^3 + 1 = x^3 + 1$ where $x = 4^2 = 16$. Now factor

$$\begin{aligned} x^3 + 1 &= (x+1)(x^2 - x + 1) \\ &= (16+1)(16^2 - 16 + 1) \\ &= 17 \cdot 241. \end{aligned}$$

So the larger prime is 241, whose digit sum is $2 + 4 + 1 = 7$.

16. Let $f(x) = |2x - 3|$. How many real solutions, x, are there to the equation $f(f(x)) = 3$?

(a) 4
(b) 3
(c) 2
(d) 1
(e) None of these

Solution. (b) The equation $f(f(x)) = 3$ is equivalent to

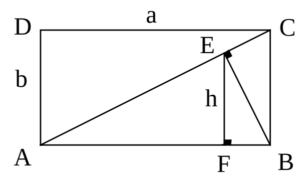
$$|2|2x - 2| - 3| = 3.$$

Our first step is to rip away the outer-most absolute value to get

$$2|2x - 3| - 3 = \pm 3.$$

This can be simplified to $|2x - 3| = 0$ or 3. In the first case, we get $2x - 3 = 0$, so $x = 3/2$. In the second case, we get $2x - 3 = \pm 3$, whose solution is $x = (3 \pm 3)/2$ which is 0 or 3. So there are 3 solutions, $x = 0, 3/2, 3$.

17. In the figure below, $ABCD$ is a rectangle, $a = DC = AB$ and $b = AD = BC$. Also, $\overline{EB} \perp \overline{AC}$ and $\overline{EF} \perp \overline{AB}$. Determine $h = EF$ as a function of a and b.



(a) $\dfrac{a^2 b}{\sqrt{a^2+b^2}}$ (b) $\dfrac{a^2 b}{a^2+b^2}$

(c) $\dfrac{ab^2}{a^2+b^2}$ (d) $\dfrac{ab^2}{\sqrt{a^2+b^2}}$

(e) None of these

Solution. (b) First we note that $\triangle EFB \sim \triangle BEC$, so

$$\frac{h}{EB} = \frac{EB}{b} \implies h = \frac{(EB)^2}{b} \quad (1)$$

Secondly, we have $\triangle BEC \sim \triangle ABC$, to obtain

$$\frac{EB}{b} = \frac{a}{AC} \implies EB = \frac{ab}{AC} \quad (2)$$

Combine equations 1 and 2 to get

$$\begin{aligned} h &= \frac{(EB)^2}{b} \\ &= \frac{(ab/AC)^2}{b} \\ &= \frac{a^2 b}{(AC)^2} \\ &= \frac{a^2 b}{a^2+b^2}, \end{aligned}$$

where the last equality was obtained by the Pythagorean Theorem applied to $\triangle ABC$.

18. Suppose Alex and Betty can mop the floor in 3 hours, Betty and Charles can mop the floor in 2 hours and Charles and Diana can mop the floor in 4 hours. How long would it take Alex and Diana to mop the floor?

(a) 6 hours (b) 8 hours

(c) 10 hours (d) 12 hours

(e) None of these

Solution. (d) Let a, b, c and d be the fraction of floor mopped by Alex, Betty, Charles and Diana respectively in one hour. Then we obtain the equations

$$\begin{aligned} a + b \qquad\qquad &= \frac{1}{3} \\ b + c \qquad &= \frac{1}{2} \\ c + d &= \frac{1}{4}. \end{aligned}$$

Now subtract the second equation from the sum of the first and third to obtain

$$a + d = \frac{1}{12}.$$

So it will take Alex and Diana 12 hours to mop the floor.

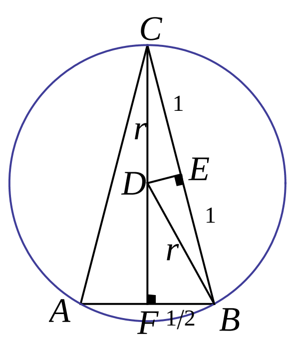
19. Determine the radius of the circumscribing circle about the isosceles triangle whose respective side lengths are 2, 2 and 1.

(a) $\frac{4}{\sqrt{15}}$ (b) $\frac{5}{\sqrt{15}}$

(c) 1 (d) $\frac{\sqrt{5}}{2}$

(e) None of these

Solution. (a) The figure below represents the triangle, $\triangle ABC$, and circumscribing circle.



$\overline{ED}$ is the perpendicular bisector of $\overline{AC}$ and $\overline{CF}$ is the perpendicular bisector of $\overline{AB}$. Thus, D is the center of the circle, and consequently $DC = DB = r$ is the radius. We first note that $\triangle CDE \sim \triangle CBF$, and so

$$\frac{r}{DE} = \frac{2}{1/2} = 4.$$

Additionally, $DE = \sqrt{r^2 - 1}$ by the Pythagorean Theorem. So

$$\begin{aligned}\frac{r}{\sqrt{r^2-1}} = 4 \quad &\Longrightarrow \quad r = 4\sqrt{r^2-1} \\ &\Longrightarrow \quad r^2 = 16(r^2-1) \\ &\Longrightarrow \quad r^2 = \frac{16}{15} \\ &\Longrightarrow \quad r = \frac{4}{\sqrt{15}}.\end{aligned}$$

20. How many solutions, $0 \le x \le 2\pi$, are there to the equation

$$\sin^2 x + \cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x + \csc^2 x = 8?$$

(a) 2

(b) 4

(c) 6

(d) 8

(e) None of these

Solution. **(d)** If we let $s^2 = \sin^2 x$, then using the standard identities and definitions, the equation becomes

$$1 + \frac{s^2}{1-s^2} + \frac{1-s^2}{s^2} + \frac{1}{1-s^2} + \frac{1}{s^2} = 8.$$

Rearrange this equation to obtain

$$(3s^2 - 2)(3s^2 - 1) = 0.$$

This leads us to

$$\sin x = \pm\sqrt{\frac{2}{3}} \quad \text{and} \quad \sin x = \pm\sqrt{\frac{1}{3}}$$

A quick glance at the plot of $y = \sin x$ for $0 \le x \le 2\pi$ instantly shows that each of these equations has 4 solutions. So there are 8 solutions in all.