# Second Annual College Problem Solving Contest
## Department of Mathematics

**Problem 1** (10 points)

Suppose that $n$ is a positive integer and there is a function $f : \{1, 2, \ldots, n\} \mapsto \mathbb{R}$ satisfying

$$f(x+y) = f(x)f(y) + 1$$

for all $1 \leq x, y \leq n$ with $1 \leq x + y \leq n$.

What is the largest possible value of $n$?

**Solution:**

By the premise, for any $1 \leq x \leq n-1$,

$$f(x+1) = f(1)f(x) + 1.$$

Let $f(1) = a$. Then $f(2) = a^2 + 1$ and $f(3) = a(a^2+1) + 1 = a^3 + a + 1$.

Assume that $n \geq 4$. Then, on one hand,

$$f(4) = f(1)f(3) + 1 = a(a^3 + a + 1) + 1 = a^4 + a^2 + a + 1$$

and, on the other hand,

$$f(4) = f(2)^2 + 1 = (a^2+1)^2 + 1 = a^4 + 2a^2 + 2.$$

Hence, we obtain the equation

$$a^4 + 2a^2 + 2 = a^4 + a^2 + a + 1,$$

or, equivalently,

$$a^2 - a + 1 = 0,$$

which has no real solutions.

Thus, the largest possible value of $n$ is 3.

**Problem 2** (10 points)

Prove that if the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

with integer coefficients assumes the value 7 for four different integer values of $x$, then it cannot take the value 14 for any integer value of $x$.

**Solution:**

Let $P(x) = 7$ at $x = a, b, c, d$ distinct integers. Then,

$$P(x) - 7 = (x - a)(x - b)(x - c)(x - d)Q(x)$$

for some polynomial $Q$ with integer coefficients.

Suppose $P(A) = 14$ for some integer $A$. Then,

$$7 = (A - a)(A - b)(A - c)(A - d)Q(A)$$

but it is impossible to write 7 as the product of five integers with four of them distinct. Hence, $P(x)$ cannot take the value 14 for any integer value of $x$.

**Problem 3** (10 points)

Prove that for every odd natural $n$, $1^n + 2^n + \ldots + n^n$ is divisible by $1 + 2 + \ldots + n$.

**Solution:**

Recall that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. Since $n$ is odd, $n + 1$ is even, thus $\frac{n+1}{2}$ is an integer. Further, $n$ and $\frac{n+1}{2}$ are relatively prime. It is therefore sufficient to prove that $1^n + 2^n + \ldots + n^n$ is divisible by both $n$ and $\frac{n+1}{2}$. We have

$$\begin{aligned}
1^n + 2^n + \ldots + n^n &\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n + \ldots + (n-1)^n + n^n \\
&\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(-\frac{n-1}{2}\right)^n + \ldots + (-1)^n + 0^n \\
&\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n - \left(\frac{n-1}{2}\right)^n + \ldots - 1^n \\
&\equiv 0 \pmod{n}
\end{aligned}$$

and

$$\begin{aligned}
1^n + 2^n + \ldots + n^n &\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n + \left(\frac{n+3}{2}\right)^n \\
&\quad + \ldots + (n-1)^n + n^n \\
&\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n + \left(-\frac{n-1}{2}\right)^n \\
&\quad + \ldots + (-2)^n + (-1)^n \\
&\equiv 1^n + 2^n + \ldots + \left(\frac{n-1}{2}\right)^n + \left(\frac{n+1}{2}\right)^n - \left(\frac{n-1}{2}\right)^n + \ldots - 2^n - 1^n \\
&\equiv \left(\frac{n+1}{2}\right)^n \pmod{n+1},
\end{aligned}$$

so

$$1^n + 2^n + \ldots + n^n \equiv 0 \left(\bmod\ \frac{n+1}{2}\right).$$

**Problem 4** (10 points)

Suppose that, for a function $f : [0, \infty) \to \mathbb{R}$,

$$\lim_{x\to\infty} \left[ f(x) + \frac{1}{|f(x)|} \right] = 0.$$

Show that the limit

$$\lim_{x\to\infty} f(x)$$

exists and evaluate it.

**Solution:**

From the premise, we infer that

$$f(x) < 0, \quad \text{for all } x \geq M, \tag{1}$$

with some $M > 0$.

Otherwise, there is a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers such that

$$x_n \to \infty,\ n \to \infty \text{ and } f(x_n) > 0,\ n = 1, 2, \ldots,$$

and hence,

$$f(x_n) + \frac{1}{f(x_n)} \geq 2,\ n = 1, 2, \ldots,$$

which is a contradiction.

Observe that the latter is easily derived from the fact that

$$\left( a - \frac{1}{a} \right)^2 \geq 0,\ a > 0.$$

For each $x \geq M$, we have:

$$\left| f(x) + \frac{1}{|f(x)|} \right| = \left| f(x) - \frac{1}{f(x)} \right| = \left| \frac{f^2(x) - 1}{f(x)} \right| = \left| \frac{(f(x) - 1)\,(f(x) + 1)}{f(x)} \right|$$

$$= \left[ 1 - \frac{1}{f(x)} \right] |f(x) + 1|$$

since, by (1), $1 - \dfrac{1}{f(x)} \geq 1$,

$$\geq |f(x) + 1| \geq 0,$$

whence, by the Squeeze Theorem, we infer that

$$\lim_{x\to\infty} |f(x) + 1| = 0,$$

i.e.,

$$\lim_{x\to\infty} f(x) = -1.$$

**Problem 5** (10 points)

2016 digits are written in a circular order. Prove that if the 2016-digit number obtained when we read these digits in clockwise direction beginning with one of the digits is divisible by 27, then if we read these digits in the same direction beginning with any other digit the new 2016-digit number that is formed is also divisible by 27.

**Solution:**

Suppose we start our number with some digit (say $a$) and read off the 2016-digit number obtained when we read these digits in clockwise direction. Let $b$ be the 2015-digit number formed by the remaining digits in order. Then, the 2016-digit number formed is:

$$M = 10^{2015} \cdot a + b$$

We will show that if $M$ is divisible by 27, then the 2016-digit number formed by starting with the next digit (after $a$, going clockwise) and ending in $a$ is also divisible by 27. This will suffice to show that the claim in the problem is true. Note that this next number (ending in $a$) is:

$$N = 10b + a$$

Now, $10M - N = (10^{2016} - 1)a$, and $10^{2016} - 1 = \underbrace{999\ldots9}_{\text{2016 nines}} = 9 \times \underbrace{111\ldots1}_{\text{2016 ones}}$. But the digits of $\underbrace{111\ldots1}_{\text{2016 ones}}$ add up to 2016 and hence, this number is divisible by 3. So, $10M - N$ is divisible by 27. Since $M$ is divisible by 27, we conclude that $N$ is also divisible by 27, and we are done!

**Problem 6** (10 points)
Prove that $\sin 1 < \log_3 \sqrt{7}$.

**Solution:**

We have $\sin 1 < \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} < \frac{7}{8}$. It remains to show that $\frac{7}{8} < \log_3 \sqrt{7}$.

The latter inequality is equivalent to $7 < \log_3 7^4$, which is true since $3^7 < 7^4$.