On the supports of periodic eigenfunctions of the Fourier Transform operator

Abstract

The $p-$ periodic generalized function

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} \gamma[n]\delta\left(x - \frac{n}{p} - mp\right) \tag{1}$$

where γ is an eigenvector of the discrete Fourier transform operator $\mathcal{F}_N$ with

$$(\mathcal{F}_N\ \gamma)[k] = \frac{1}{N}\sum_{n=0}^{N-1}\gamma[n]e^{-2\pi ikn/N} = \frac{\lambda}{\sqrt{N}}\gamma[k], \qquad k = 0, 1, \ldots, N-1,$$

and where $\lambda \in \{-1, 1, -i, i\}$ and $p = \sqrt{N}$ is an eigenfunction of the Fourier transform operator, i.e.,

$$\mathcal{F}f(x) = \lambda f(x)$$

The support of f is the lattice

$$\mathcal{L} = \{mp : m \in \mathbb{Z}\} \cup \left\{\frac{1}{p} + mp : m \in \mathbb{Z}\right\} \cup \left\{\frac{2}{p} + mp : m \in \mathbb{Z}\right\} \cup \ldots \cup \left\{\frac{p^2-1}{p} + mp : m \in \mathbb{Z}\right\}$$

and at a lattice point $x_0 = \dfrac{n}{p} + m_0 p$, $0 \leq n \leq p^2 - 1$, $m_0 \in \mathbb{Z}$, the weight of the Dirac $\delta-$ spike is $\gamma[n]$.

We will prove that in $\mathbb{R}$, the only eigenfunctions with the representation (1) such that $\gamma[n]$ is constant for $0 \leq n \leq p^2 - 1$ have for support the set of integers $\mathbb{Z}$. We will generalize this result on $\mathbb{R}^2$ and $\mathbb{R}^3$.