Solution to Problem of the Month, October 2010

Let the original check be for x dollars and y cents.

Then, Mr. Smith received y dollars and x cents from the teller.

So, from given information,

$$(100y + x) - 68 = 2\cdot(100x + y)$$

$$\Rightarrow \quad 199x - 98y = -68 \qquad \text{--- (1)}$$

$\gcd(199, 98) = 1$ which divides -68.

So, we are guaranteed to have a solution to the Diophantine equation above.

Euclid's Division Algorithm yields:

$$199 = 2\cdot(98) + 3$$
$$98 = 32\cdot(3) + 2$$
$$3 = 1\cdot(2) + 1$$

Retracing we get,

$$1 = 3 - 1\cdot(2)$$
$$= 3 - 1\cdot(98 - 32\cdot(3))$$
$$= 33\cdot(3) - 1\cdot(98)$$

So, $1 = 33\cdot(199 - 2\cdot(98)) - 1\cdot(98)$

$$\Rightarrow 1 = 33\cdot(199) - 67\cdot(98)$$
$$\Rightarrow -68 = [33\cdot(-68)]\,199 - [67\cdot(-68)]\,98$$

So, $x = (33)\cdot(-68)$ and $y = (67)\cdot(-68)$

is a solution to equation (1)

and in general,

$$x = (33)(-68) - 98t$$
$$y = (67)(-68) - 199t$$

is a solution to equation (1)

Since we are looking for the smallest check value, we must have,

$$x > 0 \quad \text{and} \quad y > 0$$

$$\Rightarrow (33)(-68) - 98t > 0 \quad \text{and} \quad (67)(-68) - 199t > 0$$

$$\Rightarrow t < -22.897 \quad \text{and} \quad t < -22.894$$

i.e. $t = -23$ is the first value of t that will work.

$$\Rightarrow x = (33)(-68) - 98(-23) = 10$$
$$y = (67)(-68) - 199(-23) = 21$$

Thus, the smallest check value is **\$10.21**