Solution to Problem of the Month May 2010

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$$

Set $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \qquad \left(1+\tan^2\theta = \sec^2\theta\right)$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$$

$$\left(\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right)$$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$\left(\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right)$$

$$\Rightarrow I = \int_0^{\pi/4} \left[\log 2 - \log(1+\tan\theta)\right] d\theta$$

$$\Rightarrow 2I = \frac{\pi}{4}\log 2 \Rightarrow \boxed{I = \frac{\pi}{8}\log 2}$$