Solution to Problem of the Month

March 2010

Let

$$S(x) = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \dots + 1001x^{1000}$$

$$\Rightarrow \left(\frac{x}{1+x}\right) S(x) = x(1+x)^{999} + 2x^2(1+x)^{998} + \dots + 1000x^{1000} + \frac{1001x^{1001}}{1+x}$$

Subtracting, we get,

$$\left(1 - \frac{x}{1+x}\right) S(x) = (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \dots + x^{1000} - \frac{1001x^{1001}}{1+x}$$

$$\Rightarrow S(x) = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999} + \dots + x^{1000}(1+x) - 1001x^{1001}$$

$$\Rightarrow S(x) = (1+x)^{1001}\left[\frac{1 - \left(\frac{x}{1+x}\right)^{1002}}{1 - \frac{x}{1+x}}\right] - 1002x^{1001}$$

(Geometric Sum)

$$\Rightarrow S(x) = (1+x)^{1002} - x^{1002} - 1002x^{1001}$$

So, coefficient of x^{50} in $S(x)$ is $\boxed{{}^{1002}C_{50}}$