Solution to Problem of the Month, December '10

Problem:

Show that there is a natural number n such that $n!$ when written in decimal notation (that is, base 10) ends exactly in 2010 zeros.

Solution:

For $n!$ to end in exactly 2010 zeros, we must have 10^{2010} as the exact power of 10 that divides $n!$

Since $10 = 2 \times 5$

and 2's occur more frequently than 5's in the list of natural numbers, it suffices to count the number of 5's that divide $n!$

There is one 5 in every multiple of 5

There is one additional 5 in every multiple of 25

and so on.

Thus,

$$\#\text{ of 5's in } n! = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \cdots$$

(Here $\lfloor x \rfloor$ (floor of x) = greatest integer less than or equal to x)

So,

$$\#\text{ of 5's in } n! \leq \frac{n}{5} + \frac{n}{25} + \cdots = \frac{n}{4} \quad \text{(geometric sum)}$$

So, for $n!$ to be divisible by 10^{2010} exactly (and no higher power of 10), we must have

$$\frac{n}{4} \geq 2010$$

$$\Rightarrow n \geq 8040$$

Since

$$\#\text{ of 5's in } 8040! = \left\lfloor \frac{8040}{5} \right\rfloor + \left\lfloor \frac{8040}{25} \right\rfloor + \left\lfloor \frac{8040}{125} \right\rfloor + \left\lfloor \frac{8040}{625} \right\rfloor + \left\lfloor \frac{8040}{3125} \right\rfloor$$

$$= 2007$$

we need 3 more factors of 5.

8045 is divisible by 5 but not 25 $\Rightarrow$ 1 more factor of 5

8050 " " " 25 " " 125 $\Rightarrow$ 2 more factors of 5

Thus, 8050! will have exactly 2010 zeros in the end.

(In fact, 8051!, 8052!, 8053! and 8054! also satisfy the criteria of our problem)