CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11–12
PART I

April 20, 2024

Prepare your Scantron form:

- **Name:** Your full name
- **Subject:** Your school name
- **Test No:** Mad Hatter 11–12
- **Period:** Part 1

Rules

- Please turn off your cell phones.
- No calculators or any other devices are allowed.
- You may use scratch paper.
- There will be two parts, 30 problems each.
- You will have 2 minutes to solve each problem.
- Record your answer on the Scantron form during those 2 minutes. Problems will not be shown again.
- Each correct answer is worth 1 point. Blank or incorrect is 0 points.
- Ties, if any, will be broken at the awards ceremony.
- Let the fun begin!

Get ready!

1. If 2024 is divided into three parts proportional to 1, 2, and 3, then what is the smallest part?

   (a) 336
   (b) $337\frac{1}{3}$
   (c) $338\frac{2}{3}$
   (d) $339\frac{1}{2}$
   (e) 344

2. The product of the areas of the front, bottom, and one of the side faces of a rectangular prism is equal to which of the following?

(a) the volume of the prism
(b) half of the volume
(c) the square root of the volume
(d) the square of the volume
(e) twice the volume

3. A triathlon consists of swimming, running, and biking. The biking is three-quarters of the total distance; the running is one-fifth; and the swimming is 2 km. What is the total distance of this triathlon?

(a) 18 km

(b) 20 km

(c) 38 km

(d) 40 km

(e) 60 km

4. Points $(-10, -8)$ and $(8, 1)$ are connected by a straight line segment. Which of the following points also lies on this line segment?

(a) $(12, 3)$

(b) $(4, -1)$

(c) $(-3, -3)$

(d) $(-12, -9)$

(e) $(-6, -5)$

5. If $2^x = 10$ and $10^y = 16$, what is the value of xy?

(a) 4

(b) $\log_2 26$

(c) $\log_2 10 + \log_{10} 16$

(d) $\sqrt{26}$

(e) 6

6. Let

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2023 \cdot 2024} = \frac{m}{n}$$

where $\frac{m}{n}$ is irreducible. Find $m + n$.

(a) 2024

(b) 2025

(c) 2026

(d) 4045

(e) 4047

7. For seven consecutive years, every April 20th a gnome was born. The sum of the ages of the three youngest gnomes is 21. What is the sum of the ages of the three oldest gnomes?

(a) 27

(b) 30

(c) 33

(d) 36

(e) 39

8. What is the value of $\dfrac{\sin 20^\circ}{\cos 70^\circ}$?

(a) 0

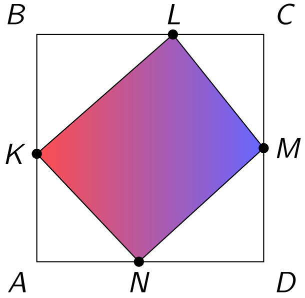
(b) $\tan 20^\circ$

(c) $\dfrac{1}{\tan 20^\circ}$

(d) $\dfrac{2}{7}$

(e) 1

9. Square $ABCD$ has area 1. Points K, L, M, and N are chosen on its sides AB, BC, CD, and DA, respectively. Which of the values $\frac{1}{10}$, $\frac{1}{2}$, and $\frac{9}{10}$ could be the area quadrilateral $KLMN$?



(a) $\frac{1}{10}$ only

(b) $\frac{1}{2}$ only

(c) $\frac{9}{10}$ only

(d) $\frac{1}{10}$ or $\frac{1}{2}$ but not $\frac{9}{10}$

(e) Any of these values could be the area

10. Simplify:

$$\frac{2024^2 - 24^2}{1524^2 - 524^2}.$$

(a) 1

(b) 2

(c) 48

(d) 500

(e) 1000

11. Three lines intersect at one point. The picture below shows the measures of angles between two pairs of these lines. What is the measure of the smallest angle between any two of these lines? (Note: the picture is not drawn to scale.)

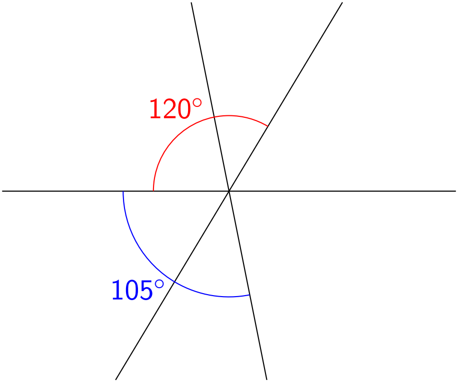
(a) 50°

(b) 45°

(c) 40°

(d) 35°

(e) 30°



12. The marked price of a book was 30% less than the suggested retail price. Alice purchased the book for half the marked price at the Fiftieth Anniversary Book sale. What percent of the suggested retail price did Alice pay?

(a) 25%

(b) 30%

(c) 35%

(d) 40%

(e) 45%

13. In $\triangle MFD$, $\angle M = 90^\circ$, $FD = 15$, and MD is twice as long as MF. What is the area of $\triangle MFD$?

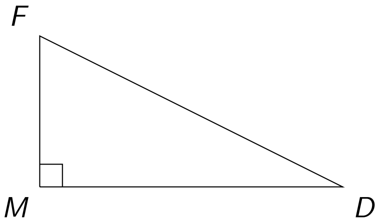
(a) 45

(b) $30\sqrt{3}$

(c) $25\sqrt{5}$

(d) $\dfrac{225}{4}$

(e) 75



14. What is the first (leftmost) digit of the smallest positive integer whose digits add up to 2024?

(a) 5

(b) 6

(c) 7

(d) 8

(e) 9

15. If the polynomial $P(x) = x^5 + ax^4 + bx^3 + 5$ is divisible by $x^2 - 1$, what is the value of $a^2 + b^2$?

(a) 13

(b) 17

(c) 20

(d) 25

(e) 26

16. How many real solutions does the equation

$$2^{x^3+1} = 8^{1-x}$$

have?

(a) 0

(b) 1

(c) 2

(d) 3

(e) infinitely many

17. Which of the following numbers is the largest?

(a) $\ln e$

(b) $\ln(\ln(e \cdot e))$

(c) $\ln\left(\sqrt{e} \cdot \ln e\right)$

(d) $\ln(\ln(e + e + e))$

(e) $\ln\left(\ln\left(\ln\left(e^{e^2}\right)\right)\right)$

18. For how many integer values of n is the number $\frac{n+1000}{n}$ also an integer?

(a) 2

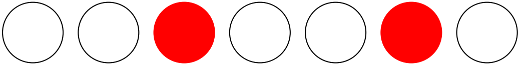
(b) 10

(c) 32

(d) 500

(e) 1000

19. Two red and five white circles are randomly arranged in a line. What is the probability that there is at least one white circle between the two red ones?



(a) $\frac{1}{6}$

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

(d) $\frac{5}{7}$

(e) $\frac{3}{4}$

20. The value of $\sqrt{20\sqrt{6}+62}$ is equal to which of the following?

(a) $\sqrt{6}+7$

(b) $10+\sqrt{6}$

(c) $3\sqrt{2}+4\sqrt{3}$

(d) $2\sqrt{3}+5\sqrt{2}$

(e) none of the above

21. How many natural numbers n have the property that the remainder of 29 upon division by n is equal to 5?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

22. If $f(x)$ is an odd function and $g(x)$ is an even function, then $f(g(x))$ is

(a) always odd

(b) always even

(c) always both odd and even

(d) never odd or even

(e) sometimes odd but never even

23. Simplify $\cot\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$.

(a) $\dfrac{1}{2\sqrt{2}}$

(b) $\dfrac{\sqrt{3}}{2}$

(c) 1

(d) $\sqrt{2}$

(e) 3

24. What are the last two digits of 24^{24}?

(a) 24

(b) 56

(c) 76

(d) 84

(e) 96

25. Let c be the largest real number such that the equation $\cos x = cx^2$ has at least three real solutions. Which inequality is true?

(a) $0.005 < c \leq 0.02$

(b) $0.02 < c \leq 0.1$

(c) $0.1 < c \leq 0.5$

(d) $0.5 < c \leq 2$

(e) $2 < c \leq 10$

26. What is the length of the shortest path that begins at a point on the circle $(x+2)^2+(y-5)^2=4$, contains at least one point on the x-axis, and ends at a point on the circle $(x-3)^2+(y-7)^2=9$?

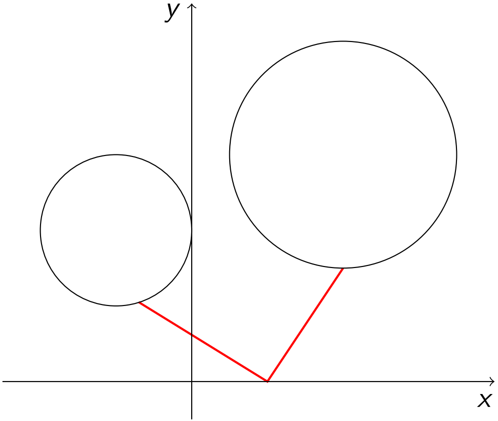
(a) 8

(b) 9

(c) 10

(d) 11

(e) 12



27. What is the smallest possible number of positive divisors of x if $x > 1$, and x, $x^{5/6}$, and $x^{7/8}$ are all integers?

(a) 25

(b) 28

(c) 35

(d) 49

(e) 63

28. Let $z = a + b\mathrm{i}$ be a complex number. If a and b are positive integers such that $a + b = 5$, what is the largest possible value of $|z|$? (Note: $|z|$ denotes the modulus of z.)

(a) $2\sqrt{2}$

(b) $2\sqrt{3}$

(c) 4

(d) $\sqrt{17}$

(e) 5

29. Both roots of the quadratic equation $x^2 - 12x + c = 0$ are prime numbers. What is the sum of digits of c?

(a) 6

(b) 7

(c) 8

(d) 9

(e) 10

30. If $S = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$ and $T = \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \dots,$

then which of the following is true?

(a) $S = 2T$

(b) $2S = 3T$

(c) $4S = 5T$

(d) $5S = 6T$

(e) $7S = 8T$

End of Part I

- Please remain seated until all Scantron forms have been collected.
- We will take a 10-minute break, then do Part II.

Stop! Answers appear on the next page.

Answers

1. b
2. d
3. d
4. b
5. a
6. e
7. c
8. e
9. e
10. b
11. b
12. c
13. a
14. d
15. e

16. b
17. a
18. c
19. d
20. d
21. e
22. b
23. a
24. c
25. b
26. a
27. a
28. d
29. c
30. e

CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11–12
PART II

April 20, 2024

Prepare your Scantron form:

- **Name:** Your full name
- **Subject:** Your school name
- **Test No:** Mad Hatter 11–12
- **Period:** Part 2

Rules

- Please turn off your cell phones.
- No calculators or any other devices are allowed.
- You may use scratch paper.
- Part II contains 30 problems.
- You will have 2 minutes to solve each problem.
- Record your answer on the Scantron form during those 2 minutes. Problems will not be shown again.
- Each correct answer is worth 1 point. Blank or incorrect is 0 points.
- Ties, if any, will be broken at the awards ceremony.
- Let the fun begin!

Get ready!

1. If we add all integers from 2001 to 2024, inclusive, and divide the result by 24, what will we get?

   (a) 2012
   (b) 2012.5
   (c) 2023
   (d) 2023.5
   (e) 2025

2. Let a be a nonzero real number. The expression

$$\frac{\sqrt{x^2+a^2} - (x-a)(x+a)/\sqrt{a^2+x^2}}{2a(a^2+x^2)}$$

can be simplified as follows:

(a) $\dfrac{a}{(x^2+a^2)^{3/2}}$

(b) $\dfrac{a}{a^2+x^2}$

(c) $\dfrac{x}{(a^2+x^2)^{3/2}}$

(d) $\dfrac{x^2}{a^3+ax^2}$

(e) 0

3. 20% of 24% of 2024 is closest to which of the following?

(a) 83

(b) 85

(c) 91

(d) 97

(e) 102

4. How many 3-digit numbers have the product of their digits equal to 20?

(a) 0

(b) 3

(c) 6

(d) 8

(e) 9

5. The digits of the seven-digit number AAABBBB add up to the two-digit number AB. What is the sum A+B?

(a) 8
(b) 9
(c) 10
(d) 11
(e) 12

6. What is the value of

$$\cos(1^\circ) + \cos(2^\circ) + \cdots + \cos(179^\circ) + \cos(180^\circ) \text{ ?}$$

(a) 1

(b) 0

(c) π

(d) 0.5

(e) -1

7. There is one value of c for which the equation $6x^2 + 4x + c = 0$ has exactly one real root (of multiplicity 2). What is this root?

(a) $-\dfrac{3}{2}$

(b) $-\dfrac{1}{3}$

(c) 0

(d) $\dfrac{2}{3}$

(e) $\dfrac{3}{4}$

8. What is the smallest possible perimeter of a rectangle with integer length and width if its area is 2024 square inches?

(a) 168 inches

(b) 172 inches

(c) 176 inches

(d) 180 inches

(e) 192 inches

9. Let $N = 3^{100} + 3^{101} + 3^{102} + 3^{103}$. How many distinct prime factors does N have?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

10. If $\tan x + \cot x = 4$, what is the value of $\sin x \cdot \cos x$?

(a) $\frac{1}{4}$

(b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2}$

(d) $\frac{1}{\sqrt{2}}$

(e) 0

11. The area of $\triangle ABC$ is 24; D is the midpoint of BC; E is the midpoint of AD; F is the midpoint of CE; and G is the midpoint of DF. What is the area of $\triangle EFG$?

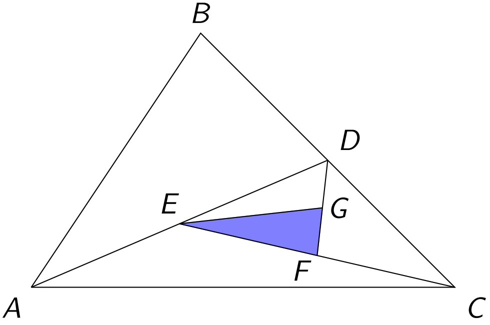
(a) 1.5

(b) 1.6

(c) 1.8

(d) 2

(e) 2.4



12. For any number N, define $\pi(N)$ to be the number of prime numbers less than or equal to N. What is $\pi(2500) - \pi(2490)$?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

13. Two freshmen, two sophomores, two juniors, and two seniors are randomly divided into two groups of four people. What is the probability that each group contains one freshman, one sophomore, one junior, and one senior?

(a) $\frac{1}{16}$

(b) $\frac{1}{8}$

(c) $\frac{1}{5}$

(d) $\frac{2}{9}$

(e) $\frac{8}{35}$

14. Each of the numbers

$$1, 2, 3, \ldots, 11, 12$$

is used exactly once as a numerator or denominator to make six fractions. (For example, we could make the following fractions: $\frac{5}{4}$, $\frac{9}{6}$, $\frac{12}{3}$, $\frac{1}{10}$, $\frac{8}{2}$, $\frac{7}{11}$.) What is the maximum possible number of integer values among these six fractions?

(a) 2

(b) 3

(c) 4

(d) 5

(e) 6

15. What is the area of a right triangle that has a hypotenuse of length 10 and one angle measuring 20°?

(a) $40 \sin 20^\circ \cos 70^\circ$

(b) $50 \sin 20^\circ$

(c) $25 \sin 40^\circ$

(d) $50 \sin^2 20^\circ$

(e) $25 \sin^2 70^\circ$

16. How many cubes have three of their eight vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$?

(a) 1

(b) 2

(c) 4

(d) 8

(e) Infinitely many

17. Alice has 24 Hershey's Kisses. In how many ways can she share them with Becky and Claire so that each of them (including Alice) has at least 5 Hershey's Kisses?

(a) 28

(b) 55

(c) 100

(d) 190

(e) 276

18. Suppose a and b are positive integers such that $a : b = 3 : 2$. What is the ratio of the least common multiple of a and b to the greatest common factor of a and b?

(a) $1 : 6$

(b) $1 : 5$

(c) $2 : 3$

(d) $5 : 1$

(e) $6 : 1$

19. How many of the numbers from 2^{20} to 2^{24}, inclusive, are divisible by 2^{18}?

(a) 16

(b) 31

(c) 32

(d) 60

(e) 61

20. Suppose $\log_x w = 4$, $\log_y w = 3$, and $\log_{xyz} w = 2$. Find $\log_z w$.

(a) -12

(b) -6

(c) 0

(d) 3

(e) 24

21. The pictures below show 7, 19, and 37 dots arranged in regular hexagons with 2, 3, and 4 dots along each side, respectively. How many dots are needed to make a regular hexagon with 10 dots along each side?

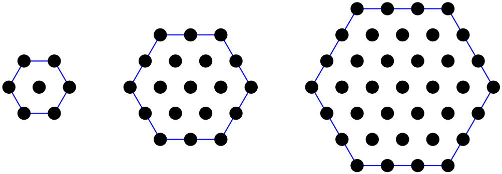
(a) 127

(b) 169

(c) 217

(d) 271

(e) 331



22. If two distinct numbers are randomly chosen from the set $\{1, 2, 3, 4, 5, 6, 7\}$, what is the probability that they are of the same parity?

(a) $\frac{2}{7}$

(b) $\frac{1}{3}$

(c) $\frac{3}{7}$

(d) $\frac{1}{2}$

(e) $\frac{2}{3}$

23. $\sqrt{1+2021\sqrt{1+2022\sqrt{1+2023\sqrt{1+2024\cdot 2026}}}} =$

(a) 2021

(b) 2022

(c) 2023

(d) 2024

(e) 2025

24. Points A, C, E, and G lie on the same line, in this order. Triangles ABC, CDE, EFG, and AHG are equilateral. The areas of the first three are 4, 16, and 25, respectively. What is the area of triangle AHG?

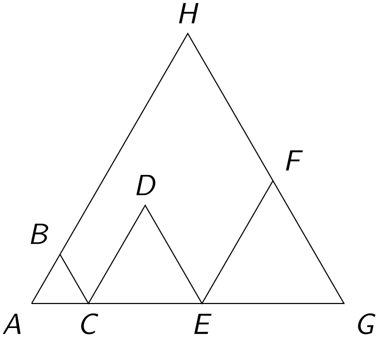
(a) 100

(b) 116

(c) 121

(d) 136

(e) 144



25. Let r, s, and t be the roots (real or complex) of the equation

$$x^3 + 3x^2 + 5x + 7 = 0.$$

Find $r^2 + s^2 + t^2$.

(a) -1

(b) 0

(c) 3

(d) 14

(e) 39

26. How many intersection points do the curves $y = x^2 - 6x$ and $y = \frac{6}{x} - 11$ have?

(a) 0

(b) 1

(c) 2

(d) 3

(e) Infinitely many

27. Over all pairs (x, y) of real numbers that satisfy $x^2 + y^2 = 9$, what is the smallest value of $\sqrt{(x-3)^2 + (y-4)^2}$?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

28. If $ABCDEFGHIJKLMNOPQRSTUVWX$ is a regular 24-gon, what is the measure of $\angle MFD$?

(a) 104°

(b) 100°

(c) 112.5°

(d) 114°

(e) 117.5°

29. Consider a sphere of radius 3 centered at the origin of the Cartesian coordinate system in the three-dimensional space. How many points on the surface of this sphere have integer coordinates?

(a) 8

(b) 12

(c) 16

(d) 24

(e) 30

30. $(\sin(15^\circ) + \cos(15^\circ))^4 =$

(a) 0.25

(b) 1

(c) 2

(d) 2.25

(e) 4

End of Part II

- Please remain seated until all Scantron forms have been collected.
- Meet your teacher, if you have arranged to do so.
- 12:00–4:00 pm: Science and Math Carnival including tables with math puzzles/games between the Science II building and the Planetarium (Note: games in Science II 207 have been canceled.)
- Approx. 2:45–3:45 pm: Awards ceremony in Science II courtyard.
  - Ties, if any, will be broken at the awards ceremony.

Stop! Answers appear on the next page.

Answers

1. b
2. a
3. d
4. e
5. c
6. e
7. b
8. d
9. c
10. a
11. a
12. a
13. e
14. d
15. c

16. b
17. b
18. e
19. e
20. a
21. d
22. c
23. b
24. c
25. a
26. d
27. b
28. c
29. e
30. d