
Solve **eight** of the following twelve problems:

1. Let f and g be real-valued functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

2. Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - y^2 + x^2y + 4$ on the set $D = \{(x,y);\ |x| \le 1,\ |y| \le 1\}$. Find the points at which these values are attained.

3. **(a)** State the Mean Value Theorem.
   (b) The following result is called Rolle's Theorem:

   > Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there must be a point $c \in (a,b)$ such that $f'(c) = 0$.

   Prove that the Mean Value Theorem and Rolle's Theorem are equivalent.

4. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.
   (a) Assume $(a_n)_{n=1}^{\infty}$ is convergent. Prove using $\varepsilon - \delta$ that (the definition of convergent sequences) $((-1)^n a_n)_{n=1}^{\infty}$ converges if and only if $\lim_{n\to\infty} a_n = 0$.
   (b) Give an example of a non-convergent sequence $(b_n)_{n=1}^{\infty}$ such that $((-1)^n b_n)_{n=1}^{\infty}$ converges.
   (c) Give an example of a non-convergent sequence $(c_n)_{n=1}^{\infty}$ such that $((-1)^n c_n)_{n=1}^{\infty}$ does not converge.

5. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $a_n > 0$ for all n, and $a_n \to a > 0$. Prove that
   $$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = a.$$

6. Suppose $p > 0$. Evaluate
   $$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} (2k)^p.$$
   Hint: Consider Darboux Sums.

Six more problems on the back!!!

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7. Use Green's theorem to evaluate the line integral $\int_{\mathcal{C}} (10xy)\ dx + (10x^2)\ dy$ along the positively oriented curve $\mathcal{C}$ consisting of the line segment from $(-3,0)$ to $(3,0)$ and the top half of the circle $x^2+y^2=9$.

8. **(a)** Define what it means for a function f to be uniformly continuous on a set E.
(b) Prove using the definition in **(a)** that

$$f_n(x) = x(\log x)^n$$

is uniformly continuous on $[0,1]$ for every $n \in \mathbb{N}$.

9. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Suppose further that for every $x \in [a,b]$ there exists a $y \in [a,b]$ such that

$$|f(y)| \leq \frac{1}{2}|f(x)|.$$

Prove that there is a point $c \in [a,b]$ such that $f(c) = 0$.
Hint: Find a sequence of points $(x_n)_{n=1}^{\infty}$ such that $f(x_n) \to 0$.

10. Let $(a_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ be sequences of positive real numbers such that

$$a_1 = b_1 = 1 \quad \text{and} \quad b_n = b_{n-1}a_n - 2 \quad \text{for} \quad n = 2,3,\ldots$$

Assume that the sequence $(b_n)_{n=1}^{\infty}$ is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

11. Let f be a continuous function on $[0,1]$. Prove that

$$\int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)\ dzdydx = \frac{1}{6}\left(\int_0^1 f(x)\ dx\right)^3$$

12. **(a)** State the definition of $L = \lim_{x\to 0^+} f(x)$. Then explain any differences between $L = \lim_{x\to 0^+} f(x)$ and $L = \lim_{x\to 0} f(x)$.
(b) Find

$$L = \lim_{x\to 0^+} (1+3x)^{1/x}$$

State clearly, and in full detail, any theorems you use.

1. Since g is continuous at 0 so
$$\lim_{x\to 0} g(x) = g(0),$$
fg is differentiable at 0 so
$$\lim_{x\to 0} \frac{f(x)g(x) - f(0)g(0)}{x} = l_1,$$
and f/g is differentiable at 0 so
$$\lim_{x\to 0} \frac{f(x)/g(x) - f(0)/g(0)}{x} = l_2,$$
and since $g(x) \neq 0$, then
$$\lim_{x\to 0} \frac{f(x)g(0) - g(x)f(0)}{x} = [g(0)]^2 \cdot l_2.$$
Now
$$[g(x) + g(0)] \cdot \frac{f(x) - f(0)}{x} = \frac{f(x)g(x) - f(0)g(0)}{x} + \frac{f(x)g(0) - g(x)f(0)}{x},$$
then take the limit as $x \to 0$ to get
$$2g(0) \cdot \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)g(x) - f(0)g(0)}{x} + \lim_{x\to 0} \frac{f(x)g(0) - g(x)f(0)}{x},$$
which yields
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \frac{l_1}{2g(0)} + \frac{g(0) \cdot l_2}{2}$$
thus f must be differentiable at 0. □

2. First, we find the critical points on the interior of this set. Setting
$$f_x = 2x + 2xy = 0 \qquad\qquad f_y = -2y + x^2 = 0$$
we obtain one solution: $x = 0$, $y = 0$, which is in D (otherwise, $x^2 = -2$, which cannot be solved in $\mathbb{R}$).
Next, we restrict our function to each segment of the boundary:
(i) $x = 1: \ f \to -y^2 + y + 5$. Its derivative is $-2y + 1$, and hence $y = 1/2$.
(ii) $x = -1$: obtain the same function, so again $y = 1/2$.
(iii) $y = 1: \ f \to 2x^2 + 3$. Its derivative is $4x$, and hence $x = 0$.
(iv) $y = -1: \ f \to 3$, and the derivative is identically equal to 0. So we have to consider all points of this segment.
Including the corners, we assemble the following list:
$$(0,0) \qquad \left(1, \frac{1}{2}\right) \qquad \left(-1, \frac{1}{2}\right) \qquad (0,1) \qquad \{(x,-1);\ |x| \le 1\} \qquad (1,1) \qquad (-1,1)$$
Evaluating the function at all of these points, we find that the maximum value $5\frac{1}{4}$ is attained at the points $\left(1, \frac{1}{2}\right)$ and $\left(-1, \frac{1}{2}\right)$, and the minimal value 3 is attained at $(0,1)$ and on the segment $\{(x,-1);\ |x| \le 1\}$. □

3. **(a)** Let f be continuous on $[a,b]$ and differentiable on (a,b). Then
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$.
(b) Assume MVT then

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

for some $c \in (a, b)$. Here using $f(a) = f(b)$.
For the converse consider $g(x) = f(x) - (x - a)\dfrac{f(b) - f(a)}{b - a}$. Since $g(a) = g(b) = f(a)$ then there is a $c \in (a, b)$ such that $g'(c) = 0$. But,

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

then $f'(c) = \frac{f(b)-f(a)}{b-a}$, for some $c \in (a, b)$. □

4. Assume that $\lim_{n\to\infty} a_n = 0$. We claim that $\lim_{n\to\infty} (-1)^n a_n = 0$. Let $\varepsilon > 0$, we want to find $N \in \mathbb{N}$ such that $|(-1)^n a_n| < \varepsilon$, for all $n > N$.
But $|(-1)^n a_n| < \varepsilon$ is equivalent to $|a_n| < \varepsilon$. Hence, the convergence of $(a_n)_{n=1}^{\infty}$ to 0 gives us the N we were looking for.
Now assume that $\lim_{n\to\infty} a_n = L \neq 0$. We know that there is an $N \in \mathbb{N}$ such that $|a_n - L| < |L|/2$, for all $n > N$. This implies that, for $n > N$, the a_n's are either all positive or all negative, and $|L|/2 \leq |a_n| \leq 3|L|/2$.
It follows that, if WLOG $a_k > 0$ then $(-1)^{k+1}a_{k+1} < 0$ and the distance between them is at least $|L|$, for all $k > N$. Hence, $((-1)^n a_n)_{n=1}^{\infty}$ diverges.
(b) Consider $(b_n)_{n=1}^{\infty}$, where $b_n = (-1)^n$. It follows that $((-1)^n b_n)_{n=1}^{\infty}$ converges, as it is the constant sequence equal to 1.
(c) Consider $(c_n)_{n=1}^{\infty}$, given by $\{1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, \cdots\}$. It follows that $((-1)^n c_n)_{n=1}^{\infty}$ is the sequence given by

$$\{1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, \cdots\}$$

which diverges. □

5. Since $a_n \to a$, given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, such that $n > N$ implies $|a_n - a| < \varepsilon$, or equivalently

$$a - \varepsilon < a_n < a + \varepsilon, \qquad n > N.$$

It follows that for $n > N$, we have

$$(a_1 a_2 \cdots a_N)(a - \varepsilon)^{n-N} < a_1 a_2 \cdots a_n < (a_1 a_2 \cdots a_N)(a + \varepsilon)^{n-N}$$

or equivalently,

$$\frac{(a_1 a_2 \cdots a_N)}{(a - \varepsilon)^N}(a - \varepsilon)^n < a_1 a_2 \cdots a_n < \frac{(a_1 a_2 \cdots a_N)}{(a + \varepsilon)^N}(a + \varepsilon)^n.$$

Taking n^{th} roots, and letting $n \to \infty$ yields

$$a - \varepsilon < \lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{\frac{1}{n}} < a + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

6. Since $x_n \to 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n| < \varepsilon$. WLOG assume that $n \geq N$ (we can do this since we want to let $n \to \infty$ anyway) and calculate

$$\begin{aligned} \left|\frac{x_1 + x_2 + \cdots + x_n}{n}\right| &\leq \left|\frac{x_1 + x_2 + \cdots + x_{N-1}}{n}\right| + \sum_{i=N}^{n} \frac{|x_i|}{n} \\ &\leq \left|\frac{x_1 + x_2 + \cdots + x_{N-1}}{n}\right| + \varepsilon\frac{(n - N)}{n}. \end{aligned}$$

Taking the limit as $n \to \infty$ of both sides gives

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} \le \varepsilon$$

Since $\varepsilon > 0$ was arbitrary the result follows.

□

7.

$$\int_C (10xy)\ dx + (10x^2)\ dy = \int\int_D 20x - 10x\ dA = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}} = 0$$

□

8. **(a)** A function f is uniformly continuous on a set E, if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $|x - y| < \delta, x, y \in E$ implies $|f(x) - f(y)| < \varepsilon$.
(b) Since products of continuous functions are continuous, $f_n(x)$ is continuous for every n. Also, continuous functions on closed and bounded intervals are uniformly continuous, and the claim follows.

□

9. Assume no such c exists. Then, by continuity, f does not change sign on $[a, b]$. WLOG assume $f(x) > 0$ for all $x \in [a, b]$. Set $x_1 = a$, and let x_n be such that $f(x_n) < \frac{1}{2} f(x_{n-1})$ for $n \ge 2$. We may assume (perhaps after selecting a subsequence) that (x_n) is convergent with limit $m \in [a, b]$. Then by construction,

$$0 < f(x_n) < \frac{1}{2^{n-1}} f(a)$$

from which we conclude that $\lim_{n\to\infty} f(x_n) = 0$. However, by continuity, we must have $0 = lim_{n\to\infty} f(x_n) = f(m)$, a contradiction. Thus a $c \in [a, b]$ with $f(c) = 0$ must exist. □

10. With $a_1 = b_1 = 1$, it follows that $b_2 = a_2 - 2,\ b_3 = a_3a_2 - 2(a_3 + 1),\ b_4 = a_4a_3a_2 - 2(a_4a_3 + a_4 + 1)$. In general

$$b_k = \Pi_{i=1}^k a_i - 2(a_k a_{k-1} \cdots a_3 + a_k a_{k-1} \cdots a_4 + \cdots + a_k + 1)$$

Now consider the partial sum

$$S_k = \sum_{n=1}^{k} \frac{1}{a_1 \cdots a_n}$$

i.e.,

$$S_k = \frac{1}{a_1} + \frac{1}{a_1a_2} + \cdots + \frac{1}{a_1 \cdots a_k} = 1 + \frac{a_k a_{k-1} \cdots a_3 + a_k a_{k-1} \cdots a_4 + \cdots + a_k + 1}{a_1 \cdots a_k}$$

so

$$b_k = \Pi_{i=1}^k a_i - 2\Pi_{i=1}^k a_i \cdot (S_k - 1) = (3 - 2S_k) \cdot \Pi_{i=1}^k a_i$$

Since $a_i > 0,\ i \ge 1$ then

$$3 - 2S_k = \frac{b_k}{\Pi_{i=1}^k a_i}$$

and

$$\lim_{k\to\infty} (3 - 2S_k) = \lim_{k\to\infty} \frac{b_k}{\Pi_{i=1}^k a_i}.$$

If $\lim_{k\to\infty} S_k = \infty$ then $\lim_{k\to\infty} \frac{b_k}{\Pi_{i=1}^k a_i} = -\infty$ which will be impossible because $a_i,\ b_i > 0,\ i \ge 1$ so $\lim_{k\to\infty} S_k < \infty$. Therefore S converges and

$$\lim_{k\to\infty} \frac{1}{\Pi_{i=1}^k a_i} = 0.$$

Moreover since the sequence (b_j) is bounded and positive there exists $M > 0$ such that $-M < 0 < b_j < M$, so

$$-M < (3 - 2S_k) \cdot \Pi_{i=1}^k a_i < M$$

or simply

$$\frac{3}{2} - \frac{M}{2\Pi_{i=1}^k a_i} < S_k < \frac{3}{2} + \frac{M}{2\Pi_{i=1}^k a_i}$$

and by the squeeze theorem

$$S = \frac{3}{2}$$

□

11. Note that the region of integration

$$S = \{(x, y, z);\ 0 \le x \le 1, x \le y \le 1, x \le z \le y\}$$

is one-sixth of the unit cube

$$R = \{(x, y, z);\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

By rearrangement of x, y, z, the other five regions may be generated. Also, $F(x, y, z) = f(x)f(y)f(z)$ is invariant to the order of x, y, z. Thus,

$$\begin{aligned} \int_0^1 \int_x^1 \int_x^y f(x)f(y)f(z)\ dz\ dy\ dx &= \frac{1}{6}\left(\int_0^1 \int_0^1 \int_0^1 f(x)f(y)f(z)\ dz\ dy\ dx\right) \\ &= \frac{1}{6}\left(\int_0^1 f(x)\ dx\right)^3 \end{aligned}$$

□

12. **(a)** $L = \lim_{x \to 0^+} f(x)$ if and only if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < x < \delta \Longrightarrow |f(x) - L| < \varepsilon$.
$L = \lim_{x \to 0} f(x)$ if and only if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|x| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.
The difference is in the types of x's that are accepted in the definition. In the first definition we have $0 < x < \delta$, and in the second having $|x| < \delta$ allows negatives x's to be considered.
(b) Using continuity of the logarithm we find

$$\ln L = \lim_{x \to 0^+} \frac{\ln(1 + 3x)}{x} = \lim_{x \to 0^+} \frac{3/(1 + 3x)}{1} = 3$$

where the limit is computed using L'Hôpital's rule. Hence,

$$L = e^3$$

□

Mathematics Department Qualifying Exam **Fall 2011**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the uniform continuity of a function $f : I \to \mathbb{R}$.

   (b) Use your definition in (a) to show that the function $f : (0,1) \to \mathbb{R}$, where $f(x) = \dfrac{1}{x^4 + 2x^2 + 1}$, is uniformly continuous .

2. Find the surface area of the part of the plane $2x + 6y + z = 10$ that lies inside of the elliptic cylinder $9x^2 + 36y^2 = 324$. Hint: The area of an ellipse is πab, where a is half the length of the major axis and b is half the length of the minor axis.

3. Let $I := [0,1]$ and let $f : I \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ 1 - x & \text{for } x \text{ irrational.} \end{cases}$$

   Show that f is injective on I and that $f(f(x)) = x$ for all $x \in I$. (Hence f is its own inverse function!) Show that f is continuous only at the point $x = \dfrac{1}{2}$.

4. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 2y^4}$ if it exists. If it does not exist, write DNE. Prove your answer.

5. Does there exist a function $f : [0,1] \to \mathbb{R}$ such that (i) f is integrable, and (ii) f has infinitely many discontinuities? If yes, exhibit such a function. If not, give a proof supporting your claim.

6. Let (x_n) be a bounded sequence and let $s = \sup\{x_n | n \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$ then there is a subsequence of (x_n) that converges to s.

Six more questions on the back !!!

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7. Let I be the set of positive integers whose digits do not contain the number 9. Show that $\sum_{n\in I} \frac{1}{n}$ is convergent.

8. Prove that $e^x > 1 + \sqrt{x} + \frac{x}{2}$ for all $x \geq 1$.

9. Suppose that $I = (0,2)$, that f is continuous at $x = 0$ and $x = 2$, and that f is differentiable on I. If $f(0) = 1$ and $f(2) = 3$, prove that $1 \in f'(I)$.

10. Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum (respectively, minimum) at an interior point c of I, show that f is not injective on I.

11. (a) Show that a differentiable function $f(x,y)$ decreases most rapidly at a point (x_0, y_0) in the direction *opposite* of the gradient $\nabla f(x_0, y_0)$.

    (b) Use your result from (a) to find the direction in which the function $f(x,y) = x^2 + y^2 - x^2y^3$ decreases fastest at the point $(2, -3)$.

12. (a) State the ε-δ definition of the limit L of a function of one variable at a point a.

    (b) Find

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}.$$

    Use your definition from (a) to prove your answer.

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   (b) Use your definition in (a) to show that the function $f : (0,1) \to \mathbb{R}$, where $f(x) = \dfrac{1}{x^4 + 2x^2 + 1}$, is uniformly continuous .

   Solution.

   (a) $f : I \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$, $x, y \in I$, then $|f(x) - f(y)| < \varepsilon$.

   (b) Let $\varepsilon > 0$ be given and set $\delta = \frac{\varepsilon}{8}$. If $x, y \in (0,1)$ and $|x - y| < \delta$, then

$$\begin{aligned}
\left| \frac{1}{x^4 + 2x^2 + 1} - \frac{1}{y^4 + 2y^2 + 1} \right| &= \left| \frac{1}{(x^2+1)^2} - \frac{1}{(y^2+1)^2} \right| \\
&= \left| \frac{(y^2+1)^2 - (x^2+1)^2}{(x^2+1)^2(y^2+1)^2} \right| \\
&= \left| \frac{(y^2 - x^2)(y^2 + x^2 + 2)}{(x^2+1)^2(y^2+1)^2} \right| \\
&< 4|(y+x)(y-x)| \\
&< 8|x - y| \\
&< 8\delta = \varepsilon.
\end{aligned}$$

2. Find the surface area of the part of the plane $2x + 6y + z = 10$ that lies inside of the elliptic cylinder $9x^2 + 36y^2 = 324$. Hint: The area of an ellipse is πab, where a is half the length of the major axis and b is half the length of the minor axis.

   Solution. A vector function for the plane is

$$\mathbf{r}(x,y) =< x, y, 10 - 2x - 6y > .$$

   Thus,

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial x} &=< 1, 0, -2 > \\
\frac{\partial \mathbf{r}}{\partial y} &=< 0, 1, -6 > \\
\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} &=< 2, 6, 1 > \\
\left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| &= \sqrt{41}
\end{aligned}$$

The surface area of a parametric surface is

$$\text{Surface Area} = \int\int_{\text{parameter space}} \left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| dA.$$

The set of possible values of x and y are all points inside of the ellipse, giving

$$\begin{aligned}\int\int_{\text{ellipse}} \sqrt{41}dA &= \sqrt{41}\int\int_{\text{ellipse}} dA \\ &= \sqrt{41}\cdot(\text{area of ellipse}) \\ &= \sqrt{41}\cdot 18\pi.\end{aligned}$$

3. Let $I := [0,1]$ and let $f : I \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ 1-x & \text{for } x \text{ irrational.} \end{cases}$$

Show that f is injective on I and that $f(f(x)) = x$ for all $x \in I$. (Hence f is its own inverse function!) Show that f is continuous only at the point $x = \frac{1}{2}$.

Solution. Let $x_1, x_2 \in I$ such that $f(x_1) = f(x_2)$.
Case 1: $x_1, x_2 \in I \cap \mathbb{Q}$ then $x_1 = x_2$.
Case 2: $x_1 \in I \cap \mathbb{Q}$ and $x_2 \in I \setminus \mathbb{Q}$ then $x_1 = 1 - x_2$ which implies that $x_2 = 1 - x_1 \in \mathbb{Q}$ a contradiction.
Case 3: $x_2 \in I \cap \mathbb{Q}$ and $x_1 \in I \setminus \mathbb{Q}$. Similar to Case 2.
Case 4: $x_1, x_2 \in I \setminus \mathbb{Q}$ then $1 - x_1 = 1 - x_2$ implying $x_1 = x_2$.
Thus f is injective on I. Note that if $x \in I \cap \mathbb{Q}$ then $f(x) \in I \cap \mathbb{Q}$ and if $x \in I \setminus \mathbb{Q}$ then $f(x) \in I \setminus \mathbb{Q}$. So

$$f(f(x)) = \begin{cases} f(x) = x, & \text{if } x \in I \cap \mathbb{Q} \\ f(1-x) = 1-(1-x) = x, & \text{if } x \in I \setminus \mathbb{Q}. \end{cases}$$

Finally, let $x \in I$ and consider an arbitrary sequence $(x_n) \subseteq I \cap \mathbb{Q}$ such that $x_n \to x$ as $n \to \infty$. Clearly $f(x_n) = x_n$ for $n \geq 1$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x$. Next consider an arbitrary sequence $(y_n) \subseteq I \setminus \mathbb{Q}$ such that $y_n \to x$ as $n \to \infty$. Clearly $f(y_n) = 1 - y_n$ and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 1 - y_n = 1 - x$. Therefore if f is continuous at x, then $x = 1 - x$ implying $x = \frac{1}{2}$.

4. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+2y^4}$ if it exists. If it does not exist, write DNE. Prove your answer.

Solution. The limit does not exist. Along the curve $y = x$, we get that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+2y^4} = \lim_{(x,x)\to(0,0)} \frac{x^2y^2}{x^4+2y^4} = \frac{1}{3}$. On the other hand, along the curve $y = 0,\ x \neq 0$, we have $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+2y^4} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$.

5. Does there exist a function $f : [0, 1] \to \mathbb{R}$ such that (i) f is integrable, and (ii) f has infinitely many discontinuities? If yes, exhibit such a function. If not, give a proof supporting your claim.

Solution. Consider the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

f has infinitely many discontinuities. However, f is integrable on $[0, 1]$. To see this, let $\varepsilon > 0$ and note that if P is any partition of $[0, 1]$, we have $L(f, P) = 0$. Thus, it remains to show that there exists a partition P of $[0, 1]$ such that $U(f, P) < \varepsilon$. To this end, note that there exists an $N \in \mathbb{N}$, such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Consider the partition

$$P = \left\{0, \frac{\varepsilon}{2}, x_{N-1}, y_{N-1}, x_{N-2}, y_{N-2}, \ldots, x_1, y_1 = 1\right\}$$

where $[x_{N-i}, y_{N-i}]$ is an interval centered at $\frac{1}{N-i}$ with length $\frac{\varepsilon}{2N\cdot K}$, and $K > 1$ is chosen so that there is no intersection between these intervals. Then, one sees that

$$U(f, P) \leq 1 \cdot \frac{\varepsilon}{2} + N\frac{\varepsilon}{2NK} < \varepsilon$$

and hence $U9f, P) - L(f, P) < \varepsilon$. We conclude that f is integrable on $[0, 1]$ with integral equal to 0.

6. Let (x_n) be a bounded sequence and let $s = \sup\{x_n | n \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$ then there is a subsequence of (x_n) that converges to s.

Solution. Recall the equivalent definition of the supremum. $s = \sup\{x_n | n \in \mathbb{N}\}$ if and only if for every $\epsilon > 0$ there exists an x_i such that $x_i > s - \epsilon$.
Now we will prove the statement by construction. Assume that $s \notin \{x_n | n \in \mathbb{N}\}$ so there are some terms of the sequence $x_i > s - 1$ for some $i \geq 1$. Call the first of these is, n_1 so $n_1 \geq 1$ and $x_{n_1} > s - 1$. Next there are again some terms of the sequence $x_i > s - \frac{1}{2}$ for some $i \geq 1$. Choose the first of these is that is greater than n_1 and call it n_2. So $n_2 > n_1$ and $x_{n_2} > s - \frac{1}{2}$. Repeat this process to get in general $n_k > n_{k-1}$ and $x_{n_k} > s - \frac{1}{k}$. Since $s > x_{n_k} > s - \frac{1}{k}$ then $x_{n_k} \to s$ as $k \to \infty$.

Six more questions on the back !!!

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7. Let I be the set of positive integers whose digits do not contain the number 9. Show that $\sum_{n\in I}\frac{1}{n}$ is convergent.

Solution. Notice that

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}<9\cdot 1 &= 9\\
\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}<9\cdot\frac{1}{10} &= \frac{9}{10}\\
&\ \vdots\\
\frac{1}{80}+\frac{1}{81}+\cdots+\frac{1}{88}<9\cdot\frac{1}{80} &= \frac{9}{80}\\
\frac{1}{100}+\frac{1}{101}+\cdots\frac{1}{108}<9\cdot\frac{1}{100} &= \frac{9}{100}\\
\frac{1}{110}+\frac{1}{111}+\cdots\frac{1}{118}<9\cdot\frac{1}{110} &= \frac{9}{110}\\
&\ \vdots
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n\in I}\frac{1}{n} &< 9+\frac{9}{10}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}\right)+\frac{9}{10}\left(\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}\right)+\cdots\\
&< 9+\frac{81}{10}+\frac{81}{100}+\frac{81}{1000}+\cdots\\
&= 9+81\sum_{k=1}^{\infty}\frac{1}{10^k}.
\end{aligned}$$

Thus we conclude that $\sum_{n\in I}\frac{1}{n}<\infty$ by the comparison test.

8. Prove that $e^x>1+\sqrt{x}+\frac{x}{2}$ for all $x\geq 1$.

Solution. We note that

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2}+\underbrace{\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots}_{>0\text{ if }x\geq 1}\\
&> 1+x+\frac{x^2}{2}\geq 1+\sqrt{x}+\frac{x}{2}\qquad (x\geq 1).
\end{aligned}$$

9. Suppose that $I=(0,2)$, that f is continuous at $x=0$ and $x=2$, and that f is differentiable on I. If $f(0)=1$ and $f(2)=3$, prove that $1\in f'(I)$.

Solution. Consider the function g defined by $g(x)=f(x)-x$. Clearly g is continuous on $[0,2]$ (continuous at 0 and 2 because f is, and continuous on I since f

is differentiable hence continuous on I). Moreover g is differentiable on I as the difference of two differentiable functions on I. Finally $g(0) = f(0) - 0 = 1 - 0 = 1$ and $g(2) = f(2) - 2 = 3 - 2 = 1$. Therefore by Rolle's theorem there exists a number $c \in I$ such that $g'(c) = 0$ that is $f'(c) - 1 = 0$ which is $f'(c) = 1$. Thus $1 \in f'(I)$.

10. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum (respectively, minimum) at an interior point c of I, show that f is not injective on I.

    Solution. Without loss of generality, assume that f has an absolute maximum at an interior point c. Let $y \in \{\max(f(a), f(b)), f(c)\}$. Applying the Intermediate Value Theorem to the function f restricted to (a, c), there exists $x_1 \in (a, c)$ such that $y = f(x_1)$. Similarly, by applying the Intermediate Value Theorem to the function f restricted to (c, b), there exists $x_2 \in (c, b)$ such that $y = f(x_2)$. So $f(x_1) = f(x_2)$ but $x_1 < x_2$. Thus f is not injective on I.

11. (a) Show that a differentiable function $f(x, y)$ decreases most rapidly at a point (x_0, y_0) in the direction *opposite* of the gradient $\nabla f(x_0, y_0)$.

    (b) Use your result from (a) to find the direction in which the function $f(x, y) = x^2 + y^2 - x^2y^3$ decreases fastest at the point $(2, -3)$.

    Solution.

    (a) Recall that $D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$. Thus the directional derivative is largest in the *negative* when $\cos\theta = -1$ or when $\vec{u}$ points in the opposite direction of ∇f.

    (b) We calculate

    $$\nabla f(2, -3) = \langle 112, -114 \rangle .$$

    It follows that f decreases fastest at $(2, -3)$ in the direction of $\vec{u} = \langle -112, 114 \rangle$.

12. (a) State the ε-δ definition of the limit L of a function of one variable at a point a.

    (b) Find

    $$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}.$$

    Use your definition from (a) to prove your answer.

    Solution.

    (a) $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$, $|f(x) - L| < \varepsilon$.

    (b) We can determine using L'Hospital's rule that the limit is equal to $\frac{3}{2}$. Thus, it remains only to show, using the definition of limits, that this is true.

Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then, for $|x-1| < \delta$,

$$\begin{aligned}
\left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| &= \left|\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} - \frac{3}{2}\right| \\
&= \left|\frac{x^2+x+1}{x+1} - \frac{3}{2}\right| \\
&= \left|\frac{2x^2-x-1}{2(x+1)}\right| \\
&= \left|\frac{(x-1)(2x+1)}{2x+2}\right| \\
&= |x-1|\left|\frac{2x+1}{2x+2}\right| \\
&< |x-1| \\
&< \varepsilon.
\end{aligned}$$

Mathematics Department Qualifying Exam **Spring 2011**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the continuity of a function $f : D \to \mathbb{R}$ at the point $x = a$.

   (b) Use your definition in (a) to show that the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{1}{\sqrt{x^2+1}}$, is continuous at $x = -1$.

2. Prove or disprove. Let $\mathbf{F} = \langle y, x \rangle$ and let A and B be the curves

$$A : y = x^2 \qquad -1 \le x \le 2$$
$$B : y = \sqrt{3(x+1)} + 1 \qquad -1 \le x \le 2$$

   Then

$$\int_A \mathbf{F} \cdot d\mathbf{r} = \int_B \mathbf{F} \cdot d\mathbf{r}.$$

3. If $|x_n| \le 2$ and $|x_{n+2} - x_{n+1}| \le \dfrac{|x_{n+1}^2 - x_n^2|}{8}$, show that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence.

4. Use the method of Lagrange multipliers to find the maximum and minimum of the function

$$f(x, y) = y^2 - 4x^2$$

   subject to the constraint

$$x^2 + 2y^2 = 4.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a one to one function. Show that there exists $c \in \mathbb{R}$ such that $f(c^2) - [f(c)]^2 < 1/4$. (*Hint:* Consider the map $t \to t - t^2$)

6. Suppose that f is integrable on $[a, b]$ such that $\displaystyle\int_a^q f(x)dx = 0$ for all rational $q \in [a, b]$. Must f be identically zero? Justify your answer.

Six more questions on the back !!!

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7. Find all points where the function $f(x) = |x|^p, p \in \mathbb{R}^+$ is differentiable. How does your answer depend on the number p?

8. Prove that
$$\overline{\lim}\, |s_n| = 0 \qquad \text{if and only if} \qquad \lim s_n = 0$$

9. Prove or disprove. If f and g are uniformly continuous functions on some interval I in $\mathbb{R}$, then $\max(f, g)$ is uniformly continuous on I.

10. Evaluate
$$\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos(x^2 y) \;\, dx dy.$$

11. (a) State the Mean Value Theorem.
    (b) Use the Mean Value theorem to prove that
$$\sqrt{1+h} < 1 + \frac{h}{2}$$
    for all $h > 0$.

12. Compute
$$\lim_{x \to 0} (\cos x)^{1/x^2}$$

Mathematics Department Qualifying Exam **Spring 2011**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the continuity of a function $f : D \to \mathbb{R}$ at the point $x = a$.

   (b) Use your definition in (a) to show that the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{1}{\sqrt{x^2+1}}$, is continuous at $x = -1$.

   Solution:

   (a) f is continuous at $a \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

   (b) $f(1) = \dfrac{1}{\sqrt{2}}$. Pick $\varepsilon > 0$. Set $\delta = \min\left(1, \dfrac{\varepsilon}{3}\right)$. Let $x \in \mathbb{R}$ such that $|x + 1| < \delta$. Then, $|x + 1| < 1$, and so $-1 < x + 1 < 1$. Therefore, $-3 < x - 1 < -1$, and so $|x - 1| < 3$. In addition.

   $$\sqrt{2}\sqrt{x^2+1}\left(\sqrt{2}+\sqrt{x^2+1}\right) > \sqrt{2}\sqrt{1}\left(\sqrt{2}+\sqrt{1}\right) > 1.$$

   Thus,

   $$\begin{aligned}
   \left|\frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}\right| &= \left|\frac{\sqrt{2}-\sqrt{x^2+1}}{\sqrt{2}\sqrt{x^2+1}} \cdot \frac{\sqrt{2}+\sqrt{x^2+1}}{\sqrt{2}+\sqrt{x^2+1}}\right| \\
   &= \left|\frac{2-(x^2+1)}{\sqrt{2}\sqrt{x^2+1}\left(\sqrt{2}+\sqrt{x^2+1}\right)}\right| \\
   &< \left|1 - x^2\right| \\
   &= |1 - x||1 + x| \\
   &< 3\delta \\
   &< \varepsilon
   \end{aligned}$$

2. Prove or disprove. Let $\mathbf{F} = \langle y, x \rangle$ and let A and B be the curves

   $$\begin{aligned}
   &A : y = x^2 && -1 \le x \le 2 \\
   &B : y = \sqrt{3(x+1)} + 1 && -1 \le x \le 2
   \end{aligned}$$

   Then

   $$\int_A \mathbf{F} \cdot d\mathbf{r} = \int_B \mathbf{F} \cdot d\mathbf{r}$$

Solution: The vector field $\mathbf{F} = \langle y, x \rangle$ is conservative (it has $f(x, y) = xy$ as a potential function). Thus the fundamental theorem of line integrals states that any two curves that start and end at the same point will have the same integral over the vector field $\mathbf{F}$.

3. If $|x_n| \le 2$ and $|x_{n+2} - x_{n+1}| \le \frac{\left|x_{n+1}^2 - x_n^2\right|}{8}$, show that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence.

Solution: First, we see that

$$\frac{\left|x_{n+1}^2 - x_n^2\right|}{8} = \frac{|x_{n+1} - x_n|\,|x_{n+1} + x_n|}{8} \le \frac{|x_{n+1} - x_n|}{2}.$$

Iterating, we see that $|x_{n+2} - x_{n+1}| \le \frac{|x_2 - x_1|}{2^n}$. Then, we can show that the sequence is Cauchy, so it converges. Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ where $N > \frac{4}{\varepsilon \ln 2}$. If $m, n > N$, and we assume without loss of generality that $n > m$, then

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m| \\
&\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\le |x_2 - x_1| \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2^{m-1}} \right) \\
&\le 4 \cdot 2^{-n} \left(2 + 2^2 + 2^3 + \cdots + 2^{n-m+1}\right) \\
&= 4 \left(2^{2-m} - 2^{-n}\right) \\
&\le \frac{4}{2^N} \\
&< \varepsilon
\end{aligned}$$

4. Use the method of Lagrange multipliers to find the maximum and minimum of the function

$$f(x, y) = y^2 - 4x^2$$

subject to the constraint

$$x^2 + 2y^2 = 4.$$

Solution: The equation $\nabla f = \lambda \nabla g$ gives two relations. Adding to these the constraint we have the following system of equations to tackle:

$$\begin{aligned}
-8x &= \lambda 2x \\
2y &= \lambda 4y \\
x^2 + 2y^2 - 4 &= 0.
\end{aligned}$$

Solving this system we see that if $x = 0$, $y = \pm\sqrt{2}$. If $y = 0$, $x = \pm 2$. Since $\nabla g \neq 0$ anywhere on the constraint, we conclude that $\lambda \neq 0$, and hence x and y can't both be non-zero simultaneously. Computing the function values we see that

$$f(0, \pm\sqrt{2}) = 2$$

and

$$f(\pm 2, 0) = -16.$$

We conclude that the minimum of the given function is -16 and its maximum is 2 on the given constraint.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a one to one function. Show that there exists $c \in \mathbb{R}$ such that $f(c^2) - [f(c)]^2 < 1/4$. (*Hint:* Consider the map $t \to t - t^2$)

Solution: Consider the function $g(t) = t - t^2$. It is clear that $g(t) \leq 1/4$, and $g(t) = 1/4$ if and only if $t = 1/2$. Observe that if $c = 0$ or $c = 1$, then $f(c^2) - [f(c)]^2 = f(c) - [f(c)]^2 = g(f(c))$. If $f(0) \neq 1/2$, then $g(f(0)) < 1/4$. If $f(0) = 1/2$, then $f(1) \neq 1/2$ since f is 1-1. Thus $g(f(1)) < 1/4$. In either case we found a $c \in \mathbb{R}$ such that $f(c^2) - [f(c)]^2 < 1/4$ as desired.

6. Suppose that f is integrable on $[a, b]$ such that $\int_a^q f(x)dx = 0$ for all rational $q \in [a, b]$. Must f be identically zero? Justify your answer.

Solution: The answer is no. Consider any function f, which is identically zero on $[a, b]$ except for finitely many points, where the function value is 1. Such a function is clearly integrable on $[a, b]$ and has integral equal to zero. In addition, it satisfies the condition that $\int_a^q f(x)dx = 0$ for all rational $q \in [a, b]$, since

$$0 \leq \int_a^q f(x)dx \leq \int_a^b f(x)dx = 0.$$

NOTE: If in addition we were to require that f be continuous on $[a, b]$, the answer would be yes, essentially by the density of $\mathbb{Q}$ in $\mathbb{R}$.

Six more questions on the back !!!

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7. Find all points where the function $f(x) = |x|^p, p \in \mathbb{R}^+$ is differentiable. How does your answer depend on the number p?

   Solution: We can rewrite our function as

$$f(x) = \begin{cases} x^p & x \geq 0 \\ (-x)^p & x < 0 \end{cases}$$

   ¿From this formulation it is clear that f is differentiable at every point in $\mathbb{R}$ except maybe at 0, since it is a monomial on the negative and positive half axes. Using the definition of the derivative at zero we see that

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^p}{h} = \lim_{h \to 0^+} h^{p-1} = 0 = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} \textbf{ only if } p > 1.$$

   hence the function is differentiable at zero only if $p > 1$.

8. Prove that

$$\overline{\lim}\, |s_n| = 0 \qquad \text{if and only if} \qquad \lim s_n = 0$$

   Solution:

$$\overline{\lim}\, |s_n| := \lim_{N \to \infty} \sup \{|s_n| : n > N\} = 0$$

   Let $\epsilon > 0$, then there exists N_0 such that whenever $n > N_0$ then $|s_n| < \epsilon$ which implies that $\lim_{n \to \infty} s_n = 0$.

   Conversely if $\lim_{n \to \infty} s_n = 0$ then $\overline{\lim}\, s_n = 0$ thus $\overline{\lim}\, |s_n| = 0$.

9. Prove or disprove. If f and g are uniformly continuous functions on some interval I in $\mathbb{R}$, then $\max(f, g)$ is uniformly continuous on I.

   Solution: Note that

$$\max(a, b) = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$$

   for all $a, b \in \mathbb{R}$. Therefore

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

   Since f, g are uniformly continuous on I, so are $f + g$ and $f - g$. Also since $f - g$ is uniformly continuous on I, so is $|f - g|$. Hence

$$\frac{1}{2}(f + g) + \frac{1}{2}|f - g| = \max(f, g)$$

   is uniformly continuous on I.

10. Evaluate

$$\int_0^{\sqrt[3]{\pi}} \int_y^{\sqrt[3]{\pi}} x^4 \cos(x^2 y) \ \ dxdy.$$

Solution: Define $R = \{(x, y) | y \leq x \leq \sqrt[3]{\pi}, 0 \leq y \leq \sqrt[3]{\pi}\}$.

Then, the given integral can be rewritten as $\iint_R x^4 \cos(x^2 y) \ \ dA$, where R is classified as a type II region.

We will change the order of integration by reclassifying R as a type I region, that is, $R = \{(x, y) | 0 \leq x \leq \sqrt[3]{\pi}, 0 \leq y \leq x\}$.

Then, we can rewrite the integral as,

$\iint_R x^4 \cos(x^2 y) \ \ dA \ = \ \int_0^{\sqrt[3]{\pi}} \int_0^x x^4 \cos(x^2 y) \ \ dydx \ = \ \frac{2}{3}$ (with a simple computation).

11. (a) State the Mean Value Theorem.

    (b) Use the Mean Value theorem to prove that

$$\sqrt{1+h} < 1 + \frac{h}{2}$$

for all $h > 0$.

Solution:

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). There there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let $x = 1 + h$. Then $h > 0$ implies that $x > 1$. Since $h = x - 1$, the desired inequality can be rewritten as $\sqrt{x} < 1 + \frac{(x-1)}{2}$; i.e.,

$$\frac{\sqrt{x} - 1}{x - 1} < \frac{1}{2}. \qquad (1)$$

Since $f(x) = \sqrt{x}$ is differentiable for $x > 0$, consider the interval $[1, x]$. Since $f(x)$ is continuous on $[1, x]$ and differentiable on $(1, x)$, we may apply the Mean Value Theorem. Thus, there exists $c \in (1, x)$ such that

$$f'(c) = \frac{f(x) - f(1)}{x - 1} = \frac{\sqrt{x} - 1}{x - 1}.$$

But, $f'(c) = \dfrac{1}{2\sqrt{c}}$. So, we have

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{1}{2\sqrt{c}} \\ &\le \frac{1}{2},\end{aligned}$$

which is (**??**).

12. Compute

$$\lim_{x\to 0} (\cos x)^{1/x^2}$$

Solution: Let

$$y = (\cos x)^{1/x^2}$$

hence

$$\ln y = \frac{\ln \cos x}{x^2}$$

Apply L'Hospital rule to get

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln \cos x}{x^2} = \lim_{x\to 0} \frac{-\tan x}{2x} = -\frac{1}{2}$$

therefore

$$\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

Mathematics Department Qualifying Exam **Fall 2010**
Subject : Analysis

Instructions : Solve ***8*** *of the following 12 problems :*

1. (a) State the ε-δ definition of the uniform continuity of a function $f : I \to \mathbb{R}$.
   (b) Use your definition in (a) to show that the function $f : (0,1) \to \mathbb{R}$, where $f(x) = \dfrac{1}{x^2+1}$, is uniformly continuous .

2. (a) Show that the maximum value of $x^2y^2z^2$ on a sphere of radius r centered at the origin is $\left(\dfrac{r^2}{3}\right)^3$.
   (b) Use (a) to show that for non-negative numbers $a, b,$ and c,
$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}.$$

3. Let $x_1 = \sin(1)$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for all $n \geq 1$.
   (a) Prove that $0 < x_n < 1$ for all $n \geq 1$.
   (b) Prove that the sequence $\langle x_n \rangle_{n\geq 1}$ converges. What is the limit?

4. Evaluate
$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dydx.$$

5. Prove that $\left\{\left(\frac{1}{n+1}, \frac{1}{n-1}\right) : n = 2, 3, 4, \ldots\right\}$ is an open covering of $(0,1)$ that does not contain a finite subcovering of $(0,1)$.

6. Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of real numbers that converges to a limit L. Prove that the series
$$\sum_{n=1}^{\infty}(a_{n+1} - 2a_n + a_{n-1})$$
converges and find its value.

   Hint: What are the partial sums of the series?

Six more questions on the back !!!

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7. Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. Find all functions $f \in \mathcal{C}^1((a,b))$ such that $f'(x) \neq 0$ for all $x \in (a,b)$ and $f'(x) = \alpha(f^{-1})'(f(x))$.

8. Consider the function
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{elsewhere} \end{cases}.$$
Prove that f is continous at $x = 0$ but discontinuous everywhere else.

9. Suppose that $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous so that
$$\alpha \int_a^c f(t)dt + \beta \int_c^b f(t)dt = 0$$
for all $c \in [a,b]$. Prove that $f(x) = 0$ for all $c \in [a,b]$.

Hint: Use the Fundamental Theorem of Calculus.

10. Let $\vec{F}, \vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be differentiable. Prove that
$$\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - (\nabla \times \vec{G}) \cdot \vec{F}.$$
Note that $\nabla \cdot \vec{F} = \mathrm{div}\vec{F}$ and $\nabla \times \vec{F} = \mathrm{curl}\,\vec{F}$.

11. (a) State the Mean Value Theorem.
    (b) Use the Mean Value Theorem to prove the following:

    Let f be differentiable on $\mathbb{R}$ such that $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Then $|f(x)| \leq |x| + 1$.

12. Suppose that f is continuous on $[a,b]$ and that
$$F(x) := \sup f([a,x]).$$
Prove that F is continuous on $[a,b]$.

Mathematics Department Qualifying Exam **Fall 2010**
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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the uniform continuity of a function $f : I \to \mathbb{R}$.

   (b) Use your definition in (a) to show that the function $f : (0,1) \to \mathbb{R}$, where $f(x) = \dfrac{1}{x^2+1}$, is uniformly continuous .

   Solution: (a) f is uniformly continuous on I if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-y| < \delta,\ x, y \in I$, then $|f(x) - f(y)| < \epsilon$.
   (b) Let $\epsilon > 0$ be given and set $\delta = \frac{\epsilon}{2}$. If $x, y \in (0,1)$ and $|x-y| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| &= \left|\frac{y^2-x^2}{(x^2+1)(y^2+1)}\right| \\ &= \left|\frac{(x+y)(y-x)}{(x^2+1)(y^2+1)}\right| \\ &< 2|x-y| \\ &< 2\delta = \epsilon.\end{aligned}$$

2. (a) Show that the maximum value of $x^2y^2z^2$ on a sphere of radius r centered at the origin is $\left(\dfrac{r^2}{3}\right)^3$.

   (b) Use (a) to show that for non-negative numbers $a, b,$ and c,

$$(abc)^{\frac{1}{3}} \le \frac{a+b+c}{3}.$$

   Solution:

   (a) We will use the method of Lagrange multipliers. Let $P(x,y,z)$ be any point on the sphere of radius r centered at the origin.
   Define $f(x,y,z,\lambda) = x^2y^2z^2 - \lambda(x^2+y^2+z^2-r^2)$. Then,
   $f_x = 2xy^2z^2 - 2\lambda x$,
   $f_y = 2x^2yz^2 - 2\lambda y$,
   $f_z = 2x^2y^2z - 2\lambda z$,
   $f_\lambda = -(x^2+y^2+z^2-r^2)$.
   Setting each to zero (ignoring trivial solutions) and solving, we get,
   $\lambda = y^2z^2$,
   $\lambda = x^2z^2$,

$\lambda = x^2y^2,$
$x^2 + y^2 + z^2 = r^2.$
This implies $x^2 = y^2 = z^2 = r^2/3$ and hence, the maximum value (by a logical reasoning argument) of $x^2y^2z^2$ is $(r^2/3)^3$.

(b) Define $a = x^2, b = y^2, c = z^2$ to be three non-negative real numbers. From (a), we have,
$abc \leq ((a+b+c)/3)^3 \Rightarrow (abc)^{1/3} \leq \dfrac{a+b+c}{3}$

3. Suppose x_n is a sequence of real numbers converging to 0. Prove that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = 0.$$

Solution: Since $x_n \to 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n| < \epsilon$. WLOG assume that $n \geq N$ (we can do this since we want to let $n \to \infty$ anyway) and calculate

$$\begin{aligned} \left|\frac{x_1 + x_2 + \cdots + x_n}{n}\right| &\leq \left|\frac{x_1 + x_2 + \cdots + x_{N-1}}{n}\right| + \sum_{i=N}^{n} \frac{|x_i|}{n} \\ &\leq \left|\frac{x_1 + x_2 + \cdots + x_{N-1}}{n}\right| + \epsilon\frac{(n-N)}{n}. \end{aligned}$$

Taking the limit as $n \to \infty$ of both sides gives

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} \leq \epsilon$$

Since $\epsilon > 0$ was arbitrary the result follows.

4. Evaluate

$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\, dydx.$$

Solution: Define $R = \{(x, y) | 0 \leq x \leq 3, \sqrt{x/3} \leq y \leq 1\}$.

Then, the given integral can be rewritten as $\displaystyle\iint_R e^{y^3}\ dA$, where R is classified as a type I region.

We will change the order of integration by reclassifying R as a type II region, that is, $R = \{(x, y) | 0 \leq x \leq 3y^2, 0 \leq y \leq 1\}$.

Then, we can rewrite the integral as,

$\displaystyle\iint_R e^{y^3}\ dA = \int_0^1 \int_0^{3y^2} e^{y^3}\ dxdy = e - 1$ (with a simple computation).

5. Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies $0 < c \leq f(x)$ for all $x \in [a, b]$ and some $c > 0$.

   (i) Prove that if f is integrable on $[a, b]$, then so is $\sqrt{f}$.

   (ii) Show that the converse is false, i.e. give an example of a function f and an interval $[a, b]$ such that $\sqrt{f}$ is integrable on $[a, b]$ but f is not.

6. Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of real numbers that converges to a limit L. Does the series

$$\sum_{n=1}^{\infty}(a_{n+1} - 2a_n + a_{n-1})$$

   converge? If so, what is its value? Justify your answer.

   Solution: Consider the partial sum

$$\begin{aligned}\sum_{n=1}^{N}(a_{n+1} - 2a_n + a_{n-1}) &= \sum_{n=1}^{N}(a_{n+1} - a_n + a_{n-1} - a_n) \\ &= \sum_{n=1}^{N}(a_{n+1} - a_n) + \sum_{n=1}^{N}(a_{n-1} - a_n) \\ &= a_{N+1} - a_1 + a_0 - a_N\end{aligned}$$

   so

$$\begin{aligned}\sum_{n=1}^{\infty}(a_{n+1} - 2a_n + a_{n-1}) &= \lim_{N\to\infty}\sum_{n=1}^{N}(a_{n+1} - 2a_n + a_{n-1}) \\ &= \lim N \to \infty(a_{N+1} - a_1 + a_0 - a_N) \\ &= L - a_1 + a_0 - L \\ &= a_0 - a_1.\end{aligned}$$

Six more questions on the back !!!

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7. Suppose $a < b$ and $a, b \in \mathbb{R}$. Find all functions $f : (a, b) \to \mathbb{R}$ such that $f'(x) \neq 0$ for all $x \in (a, b)$ and $f'(x) = \alpha(f^{-1})'(f(x))$.

   Solution: We are looking for all functions f such that $f'(x) \neq 0$ and $f'(x) = \alpha(f^{-1})'(f(x))$ for all $x \in (a, b)$. Since f is continuous and $1-1$ we conclude that f is either strictly increasing or strictly decreasing. We use that inverse function theorem to get

$$f'(x) = \frac{\alpha}{f'(f^{-1}(f(x))} = \frac{\alpha}{f'(x)}.$$

   It follows that $f'(x) = \sqrt{\alpha}$, or $f(x) = \sqrt{\alpha}x + k$ where k is any constant.

8. Find all points in $\mathbb{R}$ where the function

$$f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \\ 0 & x \text{ else} \end{cases}$$

   is differentiable. Justify your answer. Do the same for f^2.

9. Suppose that $\alpha, \beta \in R$ such that $\alpha \neq \beta$. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous so that

$$\alpha \int_a^c f(t)dt + \beta \int_c^b f(t)dt = 0$$

   for all $c \in [a, b]$. Prove that $f(x) = 0$ for all $c \in [a, b]$.

   Solution: We have $\alpha \neq \beta$. Define

$$F(x) = \alpha \int_a^x f(t)dt + \beta \int_x^b f(t)dt = \alpha \int_a^x f(t)dt - \beta \int_b^x f(t)dt.$$

   Using FTC we see that $F'(x) = (\alpha - \beta)f(x)$ for all $x \in [a, b]$. On the other hand $F(c) = 0$ for all $c \in (a, b)$ by hypothesis. It follows that since $\alpha \neq \beta$ we must have $f(x) = 0$ for all $x \in (a, b)$. Since f is continuous on $[a, b]$ we conclude that in fact $f(x) = 0$ for all $x \in [a, b]$.

10. A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 in. Find the dimensions of the box with maximum volume that meets this condition. (The girth is the perimeter of the smallest base of the box.)

11. (a) State the Mean Value Theorem.

    (b) Use the Mean Value Theorem to show that for all $x \in \mathbb{R}$, $|\sin x| \leq |x|$.

    Solution: (a) Let f be differentiable on an open interval I. For all $a, b \in I$ with $a \neq b$, there exists a c between a and b such that

$$\frac{f(b) - f(a)}{b - 1} = f'(c).$$

(b) Let $f(x) = \sin x$ and let $x, y \in \mathbb{R}$. We want to show that $|\sin x - \sin y| \leq |x - y|$. If $x = y$ the result is trivially true. Without loss of generality, assume that $x < y$. The function $\sin x$ is continuous on $[x, y]$ and differentiable on (x, y). So, we can apply the Mean Value Theorem to say that there exists $c \in (x, y)$ such that

$$\begin{aligned} f'(c) &= \frac{\sin x - \sin y}{x - y}; \text{ i.e.,} \\ \cos c &= \frac{\sin x - \sin y}{x - y}, \text{ so} \\ |\cos c| &= \left|\frac{\sin x - \sin y}{x - y}\right|. \end{aligned}$$

Since $|\cos c| \leq 1$, $\left|\frac{\sin x - \sin y}{x - y}\right| \leq 1$. Therefore,

$$|\sin x - \sin y| \leq |x - y|.$$

Set $y = 0$, and the conclusion follows.

12. Suppose that f is continuous on $[a, b]$ and that

$$F(x) := \sup f([a, x]).$$

Prove that F is continuous on $[a, b]$.

Solution: First let us prove that F is monotone. Indeed let $a \leq x_1 \leq x_2 \leq b$ then $[a, x_1] \subseteq [a, x_2]$. Therefore

$$f([a, x_1]) \subseteq f([a, x_2])$$

then

$$F(x_1) = \sup f([a, x_1]) \leq \sup f([a, x_2]) = F(x_2).$$

Let $x_0 \in [a, b]$, and consider a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq [a, b]$ that converges to x_0. Since f is continuous, we can choose $\epsilon > 0$ small enough to have

$$f(x_0 - \epsilon) \approx f(x_0) \approx f(x_0 + \epsilon).$$

Hence

$$F(x_0 - \epsilon) = F(x_0) = F(x_0 + \epsilon).$$

Since (x_n) converges to x_0 there exits N such that $n > N$ implies

$$x_0 - \epsilon < x_n < x_0 + \epsilon.$$

Since F is monotone then

$$F(x_0 - \epsilon) \leq F(x_n) \leq F(x_0 + \epsilon) \qquad \text{for } n > N$$

hence

$$\lim_{n\to\infty} F(x_0-\epsilon) \le \lim_{n\to\infty} F(x_n) \le \lim_{n\to\infty} F(x_0+\epsilon)$$

$$F(x_0-\epsilon) \le \lim_{n\to\infty} F(x_n) \le F(x_0+\epsilon).$$

Thus

$$\lim_{n\to\infty} F(x_n) = F(x_0).$$

Mathematics Department Qualifying Exam **Spring 2010**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the continuity of a function $f : D \to \mathbb{R}$ at the point $x = a$.

   (b) Use your definition in (a) to show that the function $\dfrac{1}{2x^2+1}$ is continuous at $x = -1$.

2. Find the volume of the region bounded above by the spherical surface $x^2+y^2+z^2 = 2$ and bounded below by the paraboloid $z = x^2 + y^2$.

3. Evaluate the following limit: $\lim_{x\to 0} \left(\dfrac{x}{\sin x}\right)^{1/x^2}$

4. Find the points on the surface
$$xy^2z = 2$$
that are closest to the origin.

5. Let s_n be a sequence defined recursively as follows: $s_1 = a, s_2 = b$ and $s_n = \dfrac{1}{2}(s_{n-1} + s_{n-2})$ for $n > 2$, and $a, b \in \mathbb{R}$. Find $\lim_{n\to\infty} s_n$.

   (*Hint:* Notice the following:
$$\begin{aligned}
s_1 &= a \\
s_2 &= a + (b-a) \\
s_3 &= a + (b-a) - \tfrac{1}{2}(b-a) \\
s_4 &= a + (b-a) - \tfrac{1}{2}(b-a) + \tfrac{1}{4}(b-a) \\
s_5 &= a + (b-a) - \tfrac{1}{2}(b-a) + \tfrac{1}{4}(b-a) - \tfrac{1}{8}(b-a)
\end{aligned}$$
   Now prove a general formula for s_n by induction.)

6. Let $g(x) = \big||x| - 1\big|$.

   (i) Find (with proof: justify your answer) all points where g is not differentiable.

   (ii) Prove (justify your answer) that g is integrable on $[-2, 2]$, and find $\displaystyle\int_{-2}^{2} g(x)dx$.

Six more questions on the back !!!

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7. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series such that $a_n \neq 0$ for all $n \geq 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{x \to 0} \left| \frac{f(x)}{x} \right|$ exists. Prove that the series $\sum_{n=1}^{\infty} f(a_n)$ is absolutely convergent.

8. Suppose that f, g are bounded, uniformly continuous functions on some set $S \subset \mathbb{R}$. Prove that $f \cdot g$ is uniformly continuous on S.

9. Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{n} & x \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right] \\ -\frac{1}{n} & x \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right] \end{cases} \qquad n = 1, 2, 3, 4, \ldots$$

Prove any way you can that f is integrable on $[0, 1]$.

10. Let f, g be functions with continuous second-order partial derivatives in the region R bounded by the piecewise smooth simple closed curve C. Use Green's first identity given below

$$\oint_C f \nabla g \cdot \mathbf{n}\, ds = \iint_R [(f)(\nabla \cdot \nabla g) + \nabla f \cdot \nabla g]\, dA$$

to prove Green's second identity

$$\oint_C (f \nabla g - g \nabla f) \cdot \mathbf{n}\, ds = \iint_R [(f)(\nabla \cdot \nabla g) - (g)(\nabla \cdot \nabla f)]\, dA.$$

11. (a) State the Mean Value Theorem.

    (b) Use the Mean Value Theorem to show that if f is continuous on $[3, 5]$ and differentiable on $(3, 5)$, and $f(3) = 6, f(5) = 10$, then for some point x_0 in the interval $(3, 5)$, the tangent line to the graph of f at x_0 passes through the origin.

    (*Hint:* Consider the function $g(x) = \frac{f(x)}{x}$.)

12. Let $f : [a, b] \to$ be continuous and strictly increasing on $[a, b]$. Prove that for any non-empty subset E of $[a, b]$, we have that $\sup f(E) = f(\sup E)$. (recall that $f(E) = \{f(x) : x \in E\}$).

Solutions to 2009 Fall Analysis Qualifying Exam questions

Problem 1.

(a) State the $\varepsilon - \delta$ definition of continuity of a function $f : D \to \mathbb{R}$ at a point a.

(b) Use your definition in (a) to show that the function $f(x) = \dfrac{x^2}{x^2+2}$ is continuous at $x = -1$.

Solution: f is continuous at $a \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x - a| < \delta$, and $x \in D$, then $|f(x) - f(a)| < \varepsilon$. Next, we note that $f(-1) = \dfrac{1}{3}$. We then calculate

$$\begin{aligned} |f(x) - f(-1)| &= \left|\frac{x^2}{x^2+2} - \frac{1}{3}\right| = \left|\frac{3x^2 - x^2 - 2}{3(x^2+2)}\right| = \left|\frac{2(x^2-1)}{3(x^2+2)}\right| \\ &= \left|\frac{2(x-1)}{3(x^2+2)}\right| |x - (-1)| \end{aligned}$$

Let $\delta = \min\{\frac{1}{2}, \varepsilon\}$. With this choice we see that if $|x - (-1)| < \delta$ then $\left|\frac{2(x-1)}{3(x^2+2)}\right| < \frac{1}{2}$, and hence $|f(x) - f(-1)| < \varepsilon$.

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{\alpha},$$

for some $\alpha > 1$. Show that f must be a constant function.

Solution: Fix $x \in \mathbb{R}$ and take $y \neq x$. Since $\alpha > 1$, dividing through by $|x - y| \neq 0$ we obtain

$$\frac{|f(x) - f(y)|}{|x - y|} < |x - y|^{\alpha - 1} \to 0$$

as $y \to x$. This means that (i) f is differentiable at $x \in \mathbb{R}$ and $f'(x) = 0$. Since $x \in \mathbb{R}$ was arbitrary, we conclude that f is a constant function.

Problem 3. Compute

$$\int_0^1 \int_{x^2}^1 \ln(1 + y^{3/2})\,dy\,dx.$$

Solution: The region of integration shown on Figure 1 is equivalent to the region

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{y}, 0 \leq y \leq 1\}$$

So

$$\int_0^1 \int_{x^2}^1 \ln(1 + y^{3/2})\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} \ln(1 + y^{3/2})\,dx\,dy = \int_0^1 \sqrt{y}\ln(1 + y^{3/2})\,dy$$

Let

$$u = 1 + y^{3/2} \quad \text{so} \quad du = \frac{3}{2}\sqrt{y}\,dy$$

and

$$\int_0^1 \sqrt{y}\ln(1 + y^{3/2})\,dy = \frac{2}{3}\int_1^2 \ln u\,du = \frac{2}{3}\left(u\ln u - u\right)\Big|_1^2 = \frac{2}{3}(2\ln 2 - 1).$$

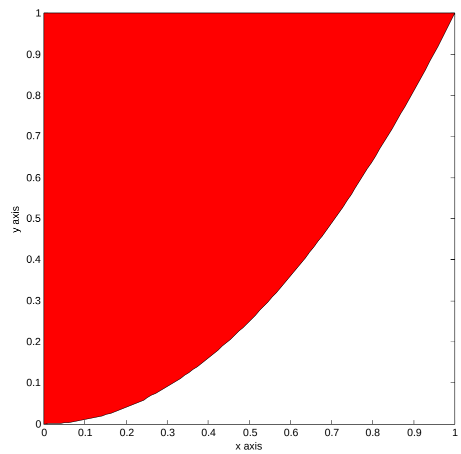

FIGURE 1. The region of integration with $0 \le x \le 1$ and $x^2 \le y \le 1$.

Problem 4. Suppose that $\{x_n\}$ satisfies

$$|x_n - x_{n+1}| < \frac{1}{(n+1)[\ln(n+1)]^2}.$$

Prove that $\{x_n\}$ is a Cauchy sequence.
Solution Note that if $n \ge m$, then

$$\begin{aligned} |x_n - x_m| &\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &< \frac{1}{n[\ln(n)]^2} + \frac{1}{(n-1)[\ln(n-1)]^2} + \cdots + \frac{1}{(m+1)[\ln(m+1)]^2} \\ &< \sum_{k=m+1}^{\infty} \frac{1}{k[\ln(k)]^2} \end{aligned}$$

Since the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k[\ln(k)]^2}$ is convergent (use the integral test) we have that given any $\varepsilon > 0$ there is an N, so that if $m \ge N$ then

$$\sum_{k=m+1}^{\infty} \frac{1}{k[\ln(k)]^2} \le \varepsilon.$$

Thus if $n, m \ge N$, then $|x_n - x_m| < \varepsilon$ and the sequence is Cauchy, as desired.

Problem 5. Find all points in $\mathbb{R}$ where the function

$$f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \\ 0 & x \quad \text{else} \end{cases}$$

is differentiable. Justify your answer. Do the same for f^2.

Solution Note first that the only rational number q so that $\sin(q) = 0$ is $q = 0$. It follows that f is not even continuous at any point other than possibly at $x = 0$. If q_n is a sequence of rational numbers converging to 0. We then have

$$\left|\frac{f(q_n) - f(0)}{q_n - 0}\right| = \left|\frac{f(q_n)}{q_n}\right| = \left|\frac{\sin q_n}{q_n}\right| \to 1 \qquad \text{as} \qquad n \to \infty.$$

If we choose an irrational sequence i_n converging to 0, we see that

$$\left|\frac{f(i_n) - f(0)}{i_n - 0}\right| = \left|\frac{f(i_n)}{i_n}\right| = 0.$$

It follows that f is not differentiable anywhere on $\mathbb{R}$. If we consider now f^2, we still have that this function is discontinuous everywhere except at 0. So we need to check differentiability on at 0. To this end note that

$$\left|\frac{f^2(x) - f^2(0)}{x - 0}\right| = \left|\frac{f^2(x)}{x}\right| \leq \left|\frac{\sin^2(x)}{x}\right| \to 0 \qquad \text{as} \qquad x \to 0.$$

Thus we conclude that f^2 is differentiable at 0 and $(f^2)'(0) = 0$.

Problem 8. Find the points on the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 1$$

where the normal line is parallel to the line that has for symmetric equations $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-0}{3}$

Solution: The ellipsoid is a level surface of the function $F(x, y, z) = x^2 + 4y^2 + 9z^2$. Let $P(x_0, y_0, z_0)$ be a point on the ellipsoid where the normal line passes. Therefore the direction of the normal line is given by $\nabla F(x_0, y_0, z_0) =< 2x_0, 8y_0, 18z_0 >$ and is parallel to the vector $< 2, -1, 3 >$ so

$$< 2x_0, 8y_0, 18z_0 >= k < 2, -1, 3 > \text{ for some scalar } k.$$

Thus

$$\begin{aligned} 2x_0 &= 2k \\ 8y_0 &= -k \\ 18z_0 &= 3k. \end{aligned}$$

Since P is on the ellipsoid then

$$k^2 + 4 \cdot \left(\frac{-k}{8}\right)^2 + 9 \cdot \left(\frac{k}{6}\right)^2 = 1$$

or

$$16k^2 + k^2 + 4k^2 = 16$$

Thus

$$k = \pm \frac{4}{\sqrt{21}}$$

so the points are

$$\left(-\frac{4}{\sqrt{21}}, \frac{1}{2\sqrt{21}}, -\frac{2}{3\sqrt{21}}\right) \text{ and } \left(\frac{4}{\sqrt{21}}, -\frac{1}{2\sqrt{21}}, \frac{2}{3\sqrt{21}}\right).$$

Problem 11. Let f, g be functions with continuous second-order partial derivatives in the region R bounded by the piecewise smooth simple closed curve C. Apply Green's theorem in vector form to show that

$$\oint_C f\nabla g \cdot \mathbf{n} \; ds = \iint_R [(f)(\nabla \cdot \nabla g) + \nabla f \cdot \nabla g] \;\; dA.$$

Solution:

Lemma 0.1. *Assume that appropriate partial derivatives exist, and let f be a scalar field and $\mathbf{F}$ be a vector field. Then*

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \tag{0.1}$$

Proof. Let $\mathbf{F} = \langle P, Q, R \rangle$.

$$\begin{aligned} \nabla \cdot (f\mathbf{F}) &= \frac{\partial (fP)}{\partial x} + \frac{\partial (fQ)}{\partial y} + \frac{\partial (fR)}{\partial z} \\ &= \left(f\frac{\partial P}{\partial x} + P\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial Q}{\partial y} + Q\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial R}{\partial z} + R\frac{\partial f}{\partial z}\right) \\ &= f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \end{aligned}$$

□

Theorem 0.2. *The following is Green's theorem in vector form:*

$$\oint_C \mathbf{F_1} \cdot \mathbf{n} \;\; ds = \iint_R \nabla \cdot \mathbf{F_1} \;\; dA. \tag{0.2}$$

Using Green's theorem and Lemma 1 above with $\mathbf{F_1} = f\nabla g$ and $\mathbf{F} = \nabla g$, we get the desired result.

Solutions to 2009 Fall Analysis Qualifying Exam questions

Problem 1.

(a) State the $\varepsilon - \delta$ definition of continuity of a function $f : D \to \mathbb{R}$ at a point a.

(b) Use your definition in (a) to show that the function $f(x) = \dfrac{x^2}{x^2+2}$ is continuous at $x = -1$.

Solution: f is continuous at $a \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x-a| < \delta$, and $x \in D$, then $|f(x) - f(a)| < \varepsilon$. Next, we note that $f(-1) = \dfrac{1}{3}$. We then calculate

$$\begin{aligned} |f(x) - f(-1)| &= \left|\frac{x^2}{x^2+2} - \frac{1}{3}\right| = \left|\frac{3x^2 - x^2 - 2}{3(x^2+2)}\right| = \left|\frac{2(x^2-1)}{3(x^2+2)}\right| \\ &= \left|\frac{2(x-1)}{3(x^2+2)}\right| |x - (-1)| \end{aligned}$$

Let $\delta = \min\{\frac{1}{2}, \varepsilon\}$. With this choice we see that if $|x-(-1)| < \delta$ then $\left|\frac{2(x-1)}{3(x^2+2)}\right| < \frac{1}{2}$, and hence $|f(x) - f(-1)| < \varepsilon$.

Problem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x-y|^\alpha,$$

for some $\alpha > 1$. Show that f must be a constant function.

Solution: Fix $x \in \mathbb{R}$ and take $y \neq x$. Since $\alpha > 1$, dividing through by $|x-y| \neq 0$ we obtain

$$\frac{|f(x) - f(y)|}{|x-y|} < |x-y|^{\alpha-1} \to 0$$

as $y \to x$. This means that (i) f is differentiable at $x \in \mathbb{R}$ and $f'(x) = 0$. Since $x \in \mathbb{R}$ was arbitrary, we conclude that f is a constant function.

Problem 3. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of integrable functions. Prove or disprove the following statements:

(i) If $f_n \to f$ pointwise, then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx$$

FALSE. To see this consider the functions

$$f_n(x) = \begin{cases} \frac{\sin(nx\pi)}{2} & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then $f_n(0) = 0$ for all n, and

$$\int_0^1 f_n(x)dx = 1 \qquad \forall n \in \mathbb{N}$$

Clearly $f_n \to 0$ point-wise on $[0,1]$ but $\int_0^1 f_n(x)dx$ do not converge to 0.

(ii) If $\int_a^b f_n(x)dx \to 0$, then $f_n \to 0$ uniformly on $[a, b]$.
FALSE. Consider $f_n(x) = x^n$ on $[0, 1]$. Then

$$\int_0^1 x^n dx = \frac{1}{n+1} \to 0 \qquad \text{as} \quad n \to \infty.$$

But the $f_n(1) = 1$ for all n! so they don't even converge pointwise to 0.

Problem 4. Suppose that $\{x_n\}$ satisfies

$$|x_n - x_{n+1}| < \frac{1}{(n+1)[\ln(n+1)]^2}.$$

Prove that $\{x_n\}$ is a Cauchy sequence.
Solution Note that if $n \geq m$, then

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &< \frac{1}{n[\ln(n)]^2} + \frac{1}{(n-1)[\ln(n-1)]^2} + \cdots + \frac{1}{(m+1)[\ln(m+1)]^2} \\ &< \sum_{k=m+1}^{\infty} \frac{1}{k[\ln(k)]^2} \end{aligned}$$

Since the series $\sum_{k=2}^{\infty} \frac{1}{k[\ln(k)]^2}$ is convergent (use the integral test) we have that given any $\varepsilon > 0$ there is an N, so that if $m \geq N$ then

$$\sum_{k=m+1}^{\infty} \frac{1}{k[\ln(k)]^2} \leq \varepsilon.$$

Thus if $n, m \geq N$, then $|x_n - x_m| < \varepsilon$ and the sequence is Cauchy, as desired.

Problem 5. Find all points in $\mathbb{R}$ where the function

$$f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \\ 0 & x \quad \text{else} \end{cases}$$

is differentiable. Justify your answer. Do the same for f^2.
Solution Note first that the only rational number q so that $\sin(q) = 0$ is $q = 0$. It follows that f is not even continuous at any point other than possibly at $x = 0$. If q_n is a sequence of rational numbers converging to 0. We then have

$$\left|\frac{f(q_n) - f(0)}{q_n - 0}\right| = \left|\frac{f(q_n)}{q_n}\right| = \left|\frac{\sin q_n}{q_n}\right| \to 1 \qquad \text{as} \qquad n \to \infty.$$

If we choose an irrational sequence i_n converging to 0, we see that

$$\left|\frac{f(i_n) - f(0)}{i_n - 0}\right| = \left|\frac{f(i_n)}{i_n}\right| = 0.$$

It follows that f is not differentiable anywhere on $\mathbb{R}$. If we consider now f^2, we still have that this function is discontinuous everywhere except at 0. So we need to check differentiability on at 0. To this end note that

$$\left|\frac{f^2(x)-f^2(0)}{x-0}\right| = \left|\frac{f^2(x)}{x}\right| \leq \left|\frac{\sin^2(x)}{x}\right| \to 0 \qquad \text{as} \qquad x \to 0.$$

Thus we conclude that f^2 is differentiable at 0 and $(f^2)'(0) = 0$.

Problem 8. Use the definition of uniform convergence of a sequence of functions to prove that $f_n(x) = \dfrac{1+2\cos^2(nx)}{\sqrt{n}}$ converges uniformly to 0 on $\mathbb{R}$.

Solution We need to check that given $\varepsilon > 0$, there is an N, so that if $n \geq N$, then $|f_n(x)-0| < \varepsilon$ for all $x \in \mathbb{R}$. Note that

$$\left|\frac{1+2\cos^2(nx)}{\sqrt{n}}\right| \leq \frac{3}{\sqrt{n}}$$

hence if $N > \frac{9}{\varepsilon^2}$, then $n \geq N$ implies $|f_n(x) - 0| < \varepsilon \ \forall x \in \mathbb{R}$. The proof is complete.

Problem 11. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

has first order partial derivatives everywhere on $\mathbb{R}^2$. Is f differentiable at $(0,0)$? Justify your answer.

Solution The existence of the partials away from $(0,0)$ is a simple calculus exercise. At $(0,0)$ we have to use the definitions. Thus we calculate

$$\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = 0$$

and

$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x}{x} = 1.$$

Thus the partial derivatives exists on all of $\mathbb{R}^2$, but they don't agree at $(0,0)$, hence our function is not differentiable at $(0,0)$.

Problem 12. Prove that the limit

$$\lim_{n\to\infty} \int_2^7 \frac{n+\cos x}{2n+\sin^2 x} dx$$

exists and find its value.

Solution We check that the integrand converges uniformly to $\dfrac{1}{2}$ on $[2,7]$:

$$\left|\frac{n+\cos x}{2n+\sin^2 x} - \frac{1}{2}\right| = \left|\frac{2\cos x - \sin^2 x}{2(2n+\sin^2 x)}\right| \leq \frac{3}{4n} \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus

$$\lim_{n\to\infty} \int_2^7 \frac{n+\cos x}{2n+\sin^2 x} dx = \int_2^7 \lim_{n\to\infty} \frac{n+\cos x}{2n+\sin^2 x} dx = \int_2^7 \frac{1}{2} dx = \frac{5}{2}$$

Mathematics Department Qualifying Exam - Solutions **Spring 2009**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of uniform continuity of a function $f : I \to \mathbb{R}$.
   Solution: f is uniformly continuous on I if and only if for all $\epsilon > 0$ there exists a $\delta > 0$, such that $|x - y| < \delta, \quad x, y \in I$ implies $|f(x) - f(y)| < \epsilon$.

   (b) Consider the function $f : (0, 1) \to \mathbb{R}$ where $f(x) = x^2$ for all $x \in (0, 1)$. Use the definition you gave in (a) to prove that f is uniformly continuous over $(0, 1)$.
   Solution: Let $\epsilon > 0$ be given and set $\delta = \dfrac{\epsilon}{2}$. If $x, y \in (0, 1)$ and $|x - y| < \delta$, then
   $$|x^2 - y^2| = |(x + y)(x - y)| < 2|x - y| < 2\delta = \epsilon,$$
   and by part (a) we conclude that $f(x) = x^2$ is uniformly continuous on $(0, 1)$.

2. Prove that the series $\displaystyle\sum_{k=1}^{+\infty} \frac{\sin(\sqrt{k}\, x)}{k^2 + x^2}$ converges uniformly over $\mathbb{R}$.
   Solution: Note that
   $$\left| \frac{\sin(\sqrt{k}\, x)}{k^2 + x^2} \right| \leq \frac{1}{k^2 + x^2} \leq \frac{1}{k^2}, \quad \forall x \in \mathbb{R}.$$
   Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < +\infty$, the Weierstrass M-test applies, and we conclude that the indicated series converges uniformly on $\mathbb{R}$.

3. Let P, Q be non-empty bounded subsets of $\mathbb{R}$ such that for each $x \in P$ there exists $y \in Q$ with $x \leq y$.

   (a) Show that $\sup(P) \leq \sup(Q)$.
   Solution: Suppose not. Then $\sup P > \sup Q$. This implies that there is an element $p \in P$ such that $\sup P \geq p > \sup Q$, which in turn gives $p > q$ for all $q \in Q$, a contradiction.

   (b) Is $\inf(P) \leq \inf(Q)$? If true, prove the statement; if false, give a counterexample.
   Solution: The answer is a resounding NO! For a counterexample, set $P = [0, 1]$ and $Q = [-2, 2]$.

4. Evaluate $\displaystyle\lim_{n \to +\infty} \int_{-2}^{1} e^{\frac{x^2}{n}}\, dx$. Justify your answer!
   Solution: The limit is equal to 3, by interchanging the limit with the integral. One

can do this, because $e^{x^2/n} \to 1$ uniformly on $[-2,1]$. To see this, let $\epsilon > 0$ be given. Then

$$\left|e^{\frac{x^2}{n}} - 1\right| \leq e^{\frac{4}{n}} - 1 < \epsilon \qquad x \in [-2,1], n \gg 1$$

since $4/n \to 0$ and the exponential function is continuous on $\mathbb{R}$.

5. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ if it exists. If it does not exist, write DNE. Prove your answer!

Solution: The limit does not exist. Along the curve $y = x^2$, we get that the quotient $\dfrac{x^2y}{x^4+y^2}$ is constant $1/2$. On the other hand

$$\lim_{(x,0)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0.$$

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   (a) At which points in $\mathbb{R}$ is f continuous? Justify your answer!

   (b) At which points in $\mathbb{R}$ is f differentiable? Justify your answer!

Solution: Standard arguments (using the definitions of continuity and differentiability) show that f is continuous only at $x = 0$, and it is actually differentiable there.

7. (a) State the Mean Value Theorem.

   Solution: Let f be differentiable on an open interval I. For all $a, b \in I$ with $a \neq b$, there exists a c between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable over $\mathbb{R}$ such that $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

   Solution: By part (a), we have

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

   for some c between x and 0. It follows that for any $x \in \mathbb{R}$ we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x) - 1}{x}\right| = |f'(c)| \leq 1,$$

and hence

$$|f(x) - 1| \leq |x|, \qquad \forall x \in \mathbb{R}.$$

Since $|f(x)| - 1 \leq |f(x) - 1|$, we obtain $|f(x)| \leq |x| + 1$, as desired.

8. Let $f : [0,1] \to \mathbb{R}$ be continuous and positive on $[0,1]$ such that $\int_0^1 f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.
Solution: Suppose not. Then there is a point $x_0 \in [0,1]$ such that $f(x_0) > 0$. By the sign preserving property of continuous functions, there exists $\epsilon > 0$, such that $f(x) > 0$ on $I \stackrel{\text{def}}{=} [x_0 - \epsilon, x_0 + \epsilon] \cap [0,1]$. Let P_I be a partition of $[0,1]$ that contains the endpoints of I. Then $L(P_I, f) > 0$, and since f is integrable on $[0,1]$, we have

$$\int_0^1 f(x)dx = L(f) = \sup_P L(P,f) \geq L(P_I, f) > 0,$$

a contradiction.

9. Put $\mathcal{C} = \left\{ \left(\frac{x}{2}, \frac{x+1}{2} \right) : 0 < x < 1 \right\}$. Show that $\mathcal{C}$ is an open cover of $(0,1)$ and that $\mathcal{C}$ does not contain a finite subcover of $(0,1)$.
Solution Given $x \in (0,1)$, it is contained in the open interval $\left(\frac{x}{2}, \frac{x+1}{2} \right)$, hence $\mathcal{C}$ is an open cover of $(0,1)$. Now if $\mathcal{C}_F$ is *any* finite subcover of $\mathcal{C}$, then there is a smallest $x_m \in (0,1)$ such that $\left(\frac{x_m}{2}, \frac{x_m+1}{2} \right) \in \mathcal{C}_F$. Consequently, $\frac{x_m}{4}$ is not in any of the sets contained in $\mathcal{C}_F$. Thus no finite subcover of $\mathcal{C}$ can be an open cover of $(0,1)$.

10. For all $x, y > 0$ we define $d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$. Is d a metric on $(0, +\infty)$? Prove your answer!
Solution: The only mildly (and even that is a stretch) interesting property to check is the triangle inequality, as $d(x,y)$ is trivially non-negative, symmetric, and 0 if and only if $x = y$. For the triangle inequality, simply calculate

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y} \right| \leq \left| \frac{1}{x} - \frac{1}{z} \right| + \left| \frac{1}{z} - \frac{1}{y} \right| = d(x,z) + d(z,y).$$

11. Let f and g be defined on $[a,b]$ with g continuous, $f \geq 0$, and f integrable. Show that there exists a point $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = g(x_0) \int_a^b f(x)dx.$$

Solution: Since g is continuous on $[a,b]$, it attains both its minimum and its maximum there. Write $g_m \stackrel{\text{def}}{=} \min_{x\in[a,b]} g(x)$ and $g_M \stackrel{\text{def}}{=} \max_{x\in[a,b]} g(x)$. Then

$$g_m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx} \le g_M$$

hence by the Intermediate Value Theorem, there is an point $x_0 \in [a,b]$, such that

$$g(x_0) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx},$$

and the proof is complete.

12. Consider the sequence $\langle a_n \rangle_{n\ge1}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \dfrac{1}{a_n} \text{ for all } n \ge 1 \end{cases}$$

Prove that the sequence $\langle a_n \rangle_{n\ge1}$ converges.
Solution: We show that a_n is bounded and monotone. We prove the first assertion by induction. We claim that $1 \le a_n \le \frac{3+\sqrt{5}}{2}$. Since $1 \le a_1 = 1 \le \frac{3+\sqrt{5}}{2}$, we have our base case. Assume now that $1 \le a_n \le \frac{3+\sqrt{5}}{2}$. Then

$$(\star)\ 1 < 2 \le a_{n+1} = 3 - \frac{1}{a_n} \le 3 - \frac{2}{3+\sqrt{5}} = \frac{7+3\sqrt{5}}{3+\sqrt{5}} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}.$$

Next we show that a_n is monotone increasing. We start by noting that $a_1 = 1 < a_2 = 2$. By the recursive formulation we see that $a_{n+1} > a_n$ if and only if

$$(\dagger)\quad a_n^2 - 3a_n + 1 < 0.$$

This happens precisely if $\frac{3-\sqrt{5}}{2} < a_n < \frac{3+\sqrt{5}}{2}$. Since $\frac{3-\sqrt{5}}{2} < 1$, the bounds exhibited in $(\star)$ assure that $(\dagger)$ holds. This completes the proof.

Mathematics Department Qualifying Exam **Fall 2009**
Subject : Analysis

Instructions : Solve ***8*** *of the following 12 problems :*

1. (a) State the ε-δ definition of the continuity of a function $f : D \to \mathbb{R}$ at a point a.

   (b) Use your definition in (a) to show that the function $\dfrac{x^2}{x^2+2}$ is continuous at $x = -1$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \le |x - y|^\alpha,$$

   for some $\alpha > 1$. Show that f must be a constant function.

3. Compute

$$\int_0^1 \int_{x^2}^1 \ln{(1 + y^{3/2})} dy dx.$$

4. Suppose that $\{x_n\}$ satisfies

$$|x_n - x_{n+1}| < \frac{1}{(n+1)[\ln(n+1)]^2}.$$

   Prove that $\{x_n\}$ is a Cauchy sequence.

5. Find all points in $\mathbb{R}$ where the function

$$f(x) = \begin{cases} \sin x, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

   is differentiable. Justify your answer. Do the same for f^2.

6. Prove that a function $f : A \to N \subset \mathbb{R}$ is uniformly continuous on A if and only if for every pair of sequences x_k and y_k in A such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$.

Six more questions on the back !!!

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7. Prove that for a function $f : [0,1] \to [0,1]$, there exists a point $x \in [0,1]$ such that $f(x) = x$.

8. Find the points on the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 1$$

where the normal line is parallel to the line that has symmetric equations $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-0}{3}$

9. (a) State the ε-δ definition of the limit L of a function of one variable at a point a.

   (b) Find

$$\lim_{x->1} \frac{x-1}{\sqrt{x}-1}.$$

   Use your definition from (a) to prove your answer.

10. (a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

   (b) Show that the converse of part (a) is false. Namely, show that a series $\sum_{k=1}^{\infty} a_k$ may have bounded partial sums and still diverge.

11. Let f, g be functions with continuous second-order partial derivatives in the region R bounded by the piecewise smooth simple closed curve C. Apply Green's theorem in vector form to show that

$$\oint_C f\nabla g \cdot \mathbf{n} ds = \iint_R [(f)(\nabla \cdot \nabla g) + \nabla f \cdot \nabla g dA.$$

12. Prove that

$$\int_0^{\infty} \frac{1}{1+x^2} \frac{x^a - x^b}{(1+x^a)(1+x^b)} dx = 0 \qquad \forall a, b \in \mathbb{R}.$$

$\left(\textit{Hint: Try the substitution } x = \dfrac{1}{u}. \right)$

Mathematics Department Qualifying Exam **Fall 2009**
Subject : Analysis – with Math 172 Material

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of the continuity of a function $f : D \to \mathbb{R}$ at a point a.

   (b) Use your definition in (a) to show that the function $\dfrac{x^2}{x^2+2}$ is continuous at $x = -1$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$|f(x) - f(y)| \leq |x - y|^\alpha,$$
for some $\alpha > 1$. Show that f must be a constant function.

3. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of integrable functions. Prove or disprove the following statements:

   (i) If $f_n \to f$ pointwise, then
$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

   (ii) If $\int_a^b f_n(x)dx \to 0$, then $f_n \to 0$ uniformly on $[a, b]$.

4. Suppose that $\{x_n\}$ satisfies
$$|x_n - x_{n+1}| < \frac{1}{(n+1)[\ln(n+1)]^2}.$$
Prove that $\{x_n\}$ is a Cauchy sequence.

5. Find all points in $\mathbb{R}$ where the function
$$f(x) = \begin{cases} \sin x, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$
is differentiable. Justify your answer. Do the same for f^2.

6. Prove that a function $f : A \to N \subset \mathbb{R}$ is uniformly continuous on A if and only if for every pair of sequences x_k and y_k in A such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$.

Six more questions on the back !!!

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7. Prove that for a function $f : [0,1] \to [0,1]$, there exists a point $x \in [0,1]$ such that $f(x) = x$.

8. Use the definition of uniform convergence of a sequence of functions to prove that $f_n(x) = \dfrac{1 + 2\cos^2(nx)}{\sqrt{n}}$ converges uniformly to 0 on $\mathbb{R}$.

9. (a) State the ε-δ definition of the limit L of a function of one variable at a point a.

   (b) Find
$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}.$$
   Use your definition from (a) to prove your answer.

10. (a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

    (b) Show that the converse of part (a) is false. Namely, show that a series $\sum_{k=1}^{\infty} a_k$ may have bounded partial sums and still diverge.

11. Prove that the limit
$$\lim_{n \to \infty} \int_2^7 \frac{n + \cos x}{2n + \sin^2 x} dx$$
exists and find its value.

12. Prove that
$$f(x) = \begin{cases} \dfrac{x^3 - xy^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$
has first order partial derivatives everywhere on $\mathbb{R}^2$. Is f differentiable at $(0,0)$? Justify your answer.

Mathematics Department Qualifying Exam **Spring 2009**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of uniform continuity of a function $f : I \to \mathbb{R}$.
   (b) Consider the function $f : (0, 1) \to \mathbb{R}$ where $f(x) = x^2$ for all $x \in (0, 1)$. Use the definition you gave in (a) to prove that f is uniformly continuous over $(0, 1)$.

2. Prove that the series $\displaystyle\sum_{k=1}^{+\infty} \frac{\sin(\sqrt{k}\ x)}{k^2 + x^2}$ converges uniformly over $\mathbb{R}$.

3. Let P, Q be non-empty bounded subsets of $\mathbb{R}$ such that for each $x \in P$ there exists $y \in Q$ with $x \le y$.
   (a) Show that $\sup(P) \le \sup(Q)$.
   (b) Is $\inf(P) \le \inf(Q)$? If true, prove the statement; if false, give a counterexample.

4. Evaluate $\displaystyle\lim_{n\to+\infty} \int_{-2}^{1} e^{\frac{x^2}{n}}\, dx$. Justify your answer!

5. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ if it exists. If it does not exist, write DNE. Prove your answer!

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ where
$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$
   (a) At which points in $\mathbb{R}$ is f continuous? Justify your answer!
   (b) At which points in $\mathbb{R}$ is f differentiable? Justify your answer!

7. (a) State the Mean Value Theorem.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable over $\mathbb{R}$ such that $f(0) = 1$ and $|f'(x)| \le 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \le |x| + 1$ for all $x \in \mathbb{R}$.

Five more questions on the back !!!

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8. Let $f : [0,1] \to \mathbb{R}$ be continuous and positive on $[0,1]$ such that $\int_0^1 f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

9. Put $\mathcal{C} = \left\{ \left(\frac{x}{2}, \frac{x+1}{2} \right) : 0 < x < 1 \right\}$. Show that $\mathcal{C}$ is an open cover of $(0,1)$ and that $\mathcal{C}$ does not contain a finite subcover of $(0,1)$.

10. For all $x, y > 0$ we define $d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$. Is d a metric on $(0, +\infty)$? Prove your answer!

11. Let f and g be defined on $[a,b]$ with g continuous, $f \geq 0$, and f integrable. Show that there exists a point $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = g(x_0) \int_a^b f(x)dx.$$

12. Consider the sequence $\langle a_n \rangle_{n \geq 1}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \dfrac{1}{a_n} \text{ for all } n \geq 1 \end{cases}$$

Prove that the sequence $\langle a_n \rangle_{n \geq 1}$ converges.

Mathematics Department Qualifying Exam **Fall 2008**
Subject : Analysis

Instructions : Solve ***8*** *of the following 12 problems :*

1. Prove that there exists a sequence $\langle n_k \rangle_{k \geq 1}$ of distinct natural numbers such that the sequence $\langle \sin(n_k) \rangle_{k \geq 1}$ converges.

2. (a) State the ϵ-δ definition of continuity of a function $f : D \to \mathbb{R}$ at the point $x = a$.

   (b) Use your definition from (a) to prove that the function $\dfrac{1}{\sqrt{2x+1}}$ is continuous at $x = 2$.

3. (a) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2008}}{1.001^n}$ converges.

   (b) Deduce that $1.001^n > n^{2008}$ if n is large enough.

4. Let A, B be non-empty subsets of $[0, +\infty)$ and $C = \{ab \,|\, a \in A \text{ and } b \in B\}$. Prove that $\inf C = \inf A \cdot \inf B$.

5. Prove that $e^x < \dfrac{1}{1-x}$ for all $0 < x < 1$.

6. (a) State the Mean Value Theorem.

   (b) Use the Mean Value Theorem to prove the following:

   Let $f : (a, b) \to \mathbb{R}$ be differentiable over (a, b). Suppose that $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant on (a, b).

7. For $n \geq 1$, put $f_n : [1, 2] \to \mathbb{R} : x \to \dfrac{nx^2 - 1}{e^x + nx^3}$

   (a) Prove that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly over $[1, 2]$.

   (b) Evaluate $\displaystyle\lim_{n \to +\infty} \int_1^2 f_n(x)\, dx$. justify your answer!

8. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \to \begin{cases} \dfrac{y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

   Is f differentiable at $(0, 0)$? Prove your answer!

Four more questions on the back !!!

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9. Prove that the equation $x^5 - x^4 + x^3 + 1 = 0$ has exactly one real solution.

10. Prove that the series $\sum_{n=1}^{+\infty} \frac{e^x - x^e}{2^n + x^2}$ converges uniformly over $[-5, 4]$.

11. Prove that the function $f : \mathbb{R} \to \mathbb{R} : x \to \frac{1}{1 + |x|}$ is uniformly continuous over $\mathbb{R}$.

12. Let $f : [a, b] \to \mathbb{R}$ be continuous and increasing on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx = f(a)(\lambda - a) + f(b)(b - \lambda)$$

for some $\lambda \in [a, b]$.

Hint : Use the Mean Value Theorem for integrals to rewrite $\int_a^b f(x)\,dx$.

Mathematics Department Qualifying Exam **Spring 2008**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. Prove the following :

$$\ln(1+x) \geq \frac{x}{x+1} \quad \text{for all } x \geq 0$$

2. (a) State the ϵ-δ definition of the limit of a function of one variable :

$$\lim_{x \to a} f(x) = L \iff \cdots$$

(b) Find $\lim_{x \to 0} \frac{x^2+2x}{x^2-2x}$. Use your definition from (a) to prove your answer.

3. Consider the function

$$f : \mathbb{R} \to \mathbb{R} : x \to \int_{x^2}^{1+x^2} 2^{t^2}\, dt$$

Calculate $f'(x)$ for all $x \in \mathbb{R}$.

4. Calculate $\lim_{n \to +\infty} \int_2^3 \frac{e^x + (-1)^n x}{1+nx^2}\, dx$. Justify your answer!

5. Let $\langle a_n \rangle_{n \geq 1}$ be a sequence of real numbers such that the sequence $\left\langle \frac{a_n+1}{a_n-1} \right\rangle$ converges to 2008. Prove that the sequence $\langle a_n \rangle_{n \geq 1}$ converges and find its limit.

6. Consider the function $f(x) = \begin{cases} xe^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Prove that f is differentiable at $x = 0$. What is $f'(0)$?

Six more questions on the back !!!

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7. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\sqrt{n}\,\cos(nx)}{x^2+n^2}$ converges uniformly over $\mathbb{R}$.

8. (a) State the ϵ-δ definition of uniform continuity of a function of one variable :

$$f \text{ is uniformly continuous over a set } D \iff \cdots$$

(b) Use your definition from (a) to prove that the function $f(x) = \dfrac{x-1}{x+1}$ is uniformly continuous over $[0, +\infty)$.

9. Let X and Y be sets and $f : X \to Y$ a function. Recall that $f(A) = \{f(a) \mid a \in A\}$ for all $A \subseteq X$. Prove that f is one-to-one on X if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

10. Let $\displaystyle\sum_{n=0}^{+\infty} a_n$ and $\displaystyle\sum_{n=0}^{+\infty} b_n$ be series of positive real numbers such that the series $\displaystyle\sum_{n=0}^{+\infty} a_n$ converges. Suppose that $\displaystyle\lim_{n\to+\infty} \frac{b_n}{a_n} = 0$. Prove that the series $\displaystyle\sum_{n=0}^{+\infty} b_n$ converges.

11. Let $\emptyset \neq A, B \subseteq \mathbb{R}$ such that $A \cup B$ is bounded above.

(a) Prove that A and B are bounded above.

(b) Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

12. Prove that $\left\{ \left(\dfrac{1}{k+1}, \dfrac{k}{k+1} \right) \middle|\, k = 2, 3, 4, \ldots \right\}$ is an open covering of $(0, 1)$ that has no finite subcover of $(0, 1)$.

Mathematics Department Qualifying Exam **Fall 2007**
Subject : Analysis

Instructions : Solve ***8*** *of the following 12 problems :*

1. Prove that $\cos^2 x \geq 1 - x^2$ for all $x \in \mathbb{R}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq 2|x-y|^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.
hint : derivatives

3. Calculate $\lim_{n \to +\infty} \int_1^2 \frac{2+nx^3}{3+nx^2}\, dx$. Justify your answer!

4. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
Is f differentiable at $(0,0)$? Prove your answer!

5. (a) State the ϵ-δ definition of continuity of a function of one variable :

$$f(x) \text{ is continuous at } x = a \iff \cdots$$

(b) Use your definition from (a) to prove that the function $f(x) = \dfrac{x}{x+1}$ is continuous at $x = -2$.

6. Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above. For $\lambda \in \mathbb{R}$, we define $\lambda S = \{\lambda s \mid s \in S\}$. Prove that $\inf(-2S) = -2 \sup S$.

7. Let $\langle x_n \rangle_{n \geq 0}$ be the sequence defined by

$$\begin{cases} x_{n+1} = n - x_n \text{ for all } n \geq 0 \\ x_0 = 0 \end{cases}$$

Prove that $x_{2k} = x_{2k+1}$ for all $k \geq 0$.

Five more questions on the back !!!

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8. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = f(b) = 0$. Prove that for all $\lambda \in \mathbb{R}$, there exists $c \in (a, b)$ such that $f'(c) = \lambda f(c)$.

hint : Use the Mean Value Theorem on a function of the form $\mu(x)f(x)$ where $\mu(x)$ is a well-chosen function.

9. Let $\langle a_n \rangle_{n\geq 0}$ be a sequence of real numbers. Put $b_n = a_n - a_{n+1}$ for all $n \geq 0$.

(a) Prove that the series $\sum_{n=0}^{+\infty} b_n$ converges if and only if the sequence $\langle a_n \rangle_{n\geq 0}$ converges.

(b) If the series $\sum_{n=0}^{+\infty} b_n$ converges, to what does it converge?

10. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n \sin(nx)}{x^n + ne^x}$ converges uniformly over $[\pi, +\infty)$.

11. Let (X, d) be a metric space. Define $D : X \times X \to \mathbb{R} : (x, y) \to \dfrac{d(x, y)}{2^{d(x,y)}}$.
Is (X, D) a metric space? Prove your answer!

12. Let $a < b < c$ and $f : (a, c) \to \mathbb{R}$ a function such that f is continuous at $x = b$ and f is uniformly continuous on (a, b) and on $[b, c)$. Prove that f is uniformly continuous on (a, c).

Name :

Spring 2007

Mathematics Department Qualifying Exam : Analysis

2/24/2007

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*Instructions : Solve **8** of the following problems :*

1. Let n be a positive integer. Prove that

$$(1-x^2)^n \geq 1 - nx^2 \quad \text{for all } x \in [0,1]$$

2. Let $\langle x_n \rangle_{n\geq 0}$ be a sequence of real numbers such that $|x_{n+1} - x_n| \leq 2^{-n}$ for all $n \in \mathbb{N}$. Prove that $\langle x_n \rangle_{n\geq 1}$ converges.

3. This exercise is about uniform continuity :

(a) State the definition of uniform continuity :

a function f is uniformly continuous on a set $D \subseteq \mathbb{R}$ if ...

(b) Use this definition to prove that the function $f(x) = x^2$ is uniformly continuous on $(0,1)$.

4. Consider the function

$$f : \mathbb{R}^2 \mapsto \mathbb{R} : (x,y) \mapsto \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Calculate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(b) Is $f(x,y)$ differentiable at $(0,0)$? Justify your answer!

5. Suppose that the series of real numbers $\sum_{k=0}^{+\infty} a_k$ converges absolutely. Prove that the series $\sum_{k=0}^{+\infty} a_k^4$ converges.

6. Let $S \subseteq \mathbb{R}$ be nonempty and bounded below. Put

$$L = \{x \in \mathbb{R} \,|\, x \text{ is a lower bound for } S\}$$

Prove that $\inf(S) = \sup(L)$.

Six more questions on the back !!!

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7. Consider the function $f_n : [1, +\infty) \mapsto \mathbb{R} : x \mapsto \dfrac{nx}{nx^2+1}$ for all $n \geq 1$.

(a) Find $\lim_{n \mapsto +\infty} f_n$ and prove that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly on $[1, +\infty)$.

(b) Evaluate $\lim_{n \mapsto +\infty} \int_1^5 f_n(x)\, dx$. Justify your answer!

8. Prove that every Cauchy sequence of real numbers is bounded WITHOUT using the fact that a Cauchy sequence of real numbers converges.

9. Prove that the series $\sum_{k=1}^{+\infty} \dfrac{x^3 - \cos(kx)}{k^2 + 3x^2}$ converges uniformly on $[-2, 1]$.

10. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuous over $\mathbb{R}$. Suppose that $f(0) = 0$. Prove that f is differentiable at $x = 0$ if and only if there exists a function $g : \mathbb{R} \mapsto \mathbb{R}$ such that g is continuous over $\mathbb{R}$ and $f(x) = x\, g(x)$ for all $x \in \mathbb{R}$.

11. Suppose that $f : [a, b] \mapsto \mathbb{Q}$ is continuous over $[a, b]$. Prove that f is constant over $[a, b]$.

12. For $n \geq 1$, put

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Prove that the sequence $\langle a_n \rangle_{n \geq 1}$ converges and that $0 \leq \lim_{n \mapsto +\infty} a_n \leq \dfrac{1}{2}$.

Mathematics Department Qualifying Exam **Fall 2006**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. Put $f : \mathbb{R} \mapsto \mathbb{R} : x \mapsto \begin{cases} x + 2x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
Show that $f'(0) = 1$.

2. (a) State the ϵ-δ definition of a limit of a function of two variables :

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \cdots$$

(b) What is $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$? Prove your answer using the definition from (a).

3. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\sqrt{n}\sin(x^n)}{x^4+n^2}$ converges uniformly over $\mathbb{R}$.

4. Let $f : \mathbb{R} \mapsto (0, +\infty)$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

(a) Find $f(0)$.

(b) Prove that $f(x-y) = \dfrac{f(x)}{f(y)}$ for all $x, y \in \mathbb{R}$.

(c) Prove that f is continuous at 0 if and only if f is continuous over $\mathbb{R}$.

5. Let $f : (a,b) \mapsto \mathbb{R}$ be twice differentiable over (a,b) and $x_1 < x_2 < x_3$ points in (a,b) with $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there exists a point $c \in (a,b)$ with $f''(c) > 0$.

6. Let $f : [a,b] \mapsto \mathbb{R}$ be continuous and positive over $[a,b]$. Prove there exists a point $c \in [a,b]$ with $f(c) = \sqrt{f(a)f(b)}$.

7. For all $n \geq 1$, let $f_n : [0,1] \mapsto \mathbb{R}$ be continuous over $[0,1]$. Suppose that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to some function f on $[0,1]$. Prove the following :

$$\forall \epsilon > 0 : \exists N \in \mathbb{N}, \exists \delta > 0 : \forall n \geq N, \forall x, y \in [0,1] : |x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$$

Five more questions on the back !!!

8.(a) For all $n \geq 1$, put $f_n : [0,1] \mapsto \mathbb{R} : x \mapsto \dfrac{n\cos(x)}{n+e^x}$. Prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly on $[0,1]$.

(b) Calculate $\displaystyle\lim_{n\to+\infty} \int_0^1 \frac{n\cos(x)}{n+e^x}\,dx$. Justify your answer!

9. (a) State the definition of a Cauchy sequence.
(b) Suppose that $0 < a < 1$ and $\langle x_n \rangle_{n\geq 1}$ is a sequence of real numbers with $|x_{n+1} - x_n| < a^n$ for all $n \geq 1$. Prove that the sequence $\langle x_n \rangle_{n\geq 1}$ converges.

10. Let (M, d) be a metric space.

(a) Let $A \subseteq M$. State the definition of an open set. So

$$A \subseteq M \text{ is open } \iff \cdots$$

(b) Let $a \in M$ and $r > 0$. Prove that the set $B(a, r) := \{x \in M \,|\, d(a, x) < r\}$ is open.

11. Let $A \subseteq \mathbb{R}$ be compact and $\lambda > 0$. We define $\lambda A := \{\lambda x \,|\, x \in A\}$. Prove that λA is compact.

hint : use the characterization of compact sets that involves sequences

12. Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded.

(a) Prove there exists a sequence $\langle s_n \rangle_{n \geq 1}$ in S that converges to $\sup S$.

(b) Is $\sup S \in S$?

Mathematics Department Qualifying Exam **Spring 2006**
Subject : Analysis

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*Instructions : Solve **8** of the following problems :*

1. Prove that the series $\sum_{n=1}^{+\infty} \frac{n\cos(nx)}{e^x + n^3}$ converges uniformly over $\mathbb{R}$.

2. (a) Let I be an interval, $f : I \mapsto \mathbb{R}$ a function and $c \in I$. State the ϵ-δ definition :

$$f \text{ is continuous at } c \text{ if and only if } \ldots$$

(b) Use the definition you gave in (a) to prove that $f(x) = \frac{x}{x^2+1}$ is continuous at $x = 1$.

3. Prove that the sequence $\left\langle \frac{n\ \sin(n)}{n+1} \right\rangle_{n\geq 1}$ has a convergent subsequence.

4. Prove that $1 + 2x\ln(x) \leq x^2$ for all $x \geq 1$.

5. (a) Calculate $\lim_{x\mapsto 0^+} \sqrt{x}\ln(x)$.

(b) Prove that $f(x) = \sqrt{x}\ln(x)$ is uniformly continuous over $(0,1)$ (do not use the definition of uniform continuity).

6. Calculate $\lim_{n\mapsto+\infty} n\int_0^{\frac{1}{n}} \cos(x^2)\,dx$. Justify your answer!

hint : use the Mean Value Theorem for integrals

7. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be differentiable over $\mathbb{R}$. Prove that f is continuous over $\mathbb{R}$.

8. For $n \geq 1$, put

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

Prove that the sequence $\langle x_n \rangle_{n\geq 1}$ converges.

hint : prove that the sequence $\langle x_n \rangle_{n\geq 1}$ is increasing and bounded above

9. Let (M,d) be a metric space. We define $d^2 : M \times M \mapsto \mathbb{R} : (x,y) \mapsto (d(x,y))^2$. Is (M,d^2) a metric space? Prove your answer!

10. Prove that $f(x) = \frac{x}{x+1}$ is uniformly continuous over $[0,+\infty)$ using the definition of uniform continuity.

Two more questions on the back !!!

11. Let $A = \mathbb{Q} \cap [0, 1]$. Is A compact? Prove your answer!
hint : use the characterization of compact sets that involves sequences!

12. Calculate $\lim\limits_{n \mapsto +\infty} \int_0^1 \frac{nx}{n + x^3}\, dx$. Justify your answer!

Mathematics Department Qualifying Exam **Fall 2005**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. Let X and Y be sets and $f : X \mapsto Y$ a function. Recall that $f(A) = \{f(a) \mid a \in A\}$ for all $A \subseteq X$. Prove that f is one-to-one on X if and only if $f(A \setminus B) = f(A) \setminus f(B)$ for all $A, B \subseteq X$.

2. Prove that $\cos(x) \geq 1 - \dfrac{x^2}{2}$ for all $x \geq 0$.

3. (a) Let $\langle \boldsymbol{a}_n \rangle_{n\geq 1}$ be a sequence in the k-dimensional Euclidean space $\mathbb{R}^k$ and $\boldsymbol{a} \in \mathbb{R}^k$. State the ε-N-definition : $\langle \boldsymbol{a}_n \rangle_{n\geq 1}$ converges to $\boldsymbol{a}$ if ...
(b) Let $\langle \boldsymbol{a}_n \rangle_{n\geq 1}$ (respectively $\langle \boldsymbol{b}_n \rangle_{n\geq 1}$) be a sequence in $\mathbb{R}^k$ that converges to $\boldsymbol{a} \in \mathbb{R}^k$ (respectively $\boldsymbol{b} \in \mathbb{R}^k$). To which element of $\mathbb{R}^k$ does the sequence $\langle 2\boldsymbol{a}_n - 3\boldsymbol{b}_n \rangle_{n\geq 1}$ converge? Prove your answer using the definition you stated in (a).

4. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose that f is continuous at $x = 0$. Prove that f is continuous over $\mathbb{R}$.

5. Let $f : (a,b) \mapsto \mathbb{R}$ be differentiable over (a,b) such that f' is bounded on (a,b). Prove that f is uniformly continuous over (a,b).

6. Let $f : [1,2] \mapsto [0,4]$ be a continuous function such that $f(1) = 0$ and $f(2) = 3$. Prove that there exists $c \in [1,2]$ such that $f(c) = c$.

7. Evaluate $\displaystyle\lim_{n\mapsto+\infty} \int_0^1 \frac{x}{x^2 + ne^x}\, dx$. Justify your answer!

8. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} e^{-nx}$ converges uniformly over $[1, +\infty)$ to some function f. Find a closed form for $f'(x)$.

9. Let (M,d) be a metric space. For $x, y \in M$ we define $e(x,y) = \min\{1, d(x,y)\}$. Prove that (M,e) is a metric space.

10. Let $\langle a_n \rangle_{n\geq 1}$ be a sequence of real numbers that converges to $\alpha \in \mathbb{R}$. Suppose that $\beta \in \mathbb{R}$ such that $a_n \leq \beta$ for all n sufficiently large. Prove that $\alpha \leq \beta$ WITHOUT using the Pinching Theorem.

11. Let $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ such that $\dfrac{a_n}{n+1} + \dfrac{a_{n-1}}{n} + \cdots + \dfrac{a_1}{2} + a_0 = 0$. Put $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Use the Mean Value Theorem (or Rolle's Theorem) to prove that $P(a) = 0$ for some $a \in (0,1)$.

12. Let $A, B \subseteq \mathbb{R}$ be compact such that $A \cap B = \emptyset$. Prove that there exists $\delta > 0$ such that $|a - b| \geq \delta$ for all $a \in A$ and all $b \in B$.
hint : put $\delta = \inf\{|a-b| \mid a \in A, b \in B\}$; for all $n \geq 1$, find $a_n \in A$ and $b_n \in B$ such that $\langle |a_n - b_n| \rangle_{n\geq 1}$ converges to δ

Mathematics Department Qualifying Exam **Spring 2005**
Subject : Analysis

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*Instructions : Solve **8** of the following problems :*

1. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be an even and monotonic function. Prove that f is constant over $\mathbb{R}$.

2. Prove the "Ratio Test" :

Let $\sum_{n=0}^{+\infty} a_n$ be a series of real numbers such that the limit $L := \lim_{n \mapsto +\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists. Then

(i) the series $\sum_{n=0}^{+\infty} a_n$ converges absolutely if $L < 1$.

(ii) the series $\sum_{n=0}^{+\infty} a_n$ diverges if $L > 1$.

3. Define the function $f : \mathbb{R} \mapsto \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

(a) Show that f is not continuous at $x = 0$.

(b) Can you alter the definition of $f(0)$ to make f continuous at $x = 0$? Justify your answer!

4. Give an ϵ-δ proof of the fact that the real function $f(x) = \frac{1}{x}$ is continuous at $x = 2$.

5. (a) Let $f : [a, b] \mapsto \mathbb{R}$ be bounded. Define what it means for f to be Riemman integrable over $[a, b]$ using the notions of upper and lower sums.

(b) Use your definition of part (a) to show that the function $f : \mathbb{R} \mapsto \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is Riemann integrable over any closed and bounded interval $[a, b]$.

6. Let $A \subseteq \mathbb{R}$ and for all $n \geq 1$, let $f_n : A \mapsto \mathbb{R}$ be a function that is uniformly continuous on A. Suppose that the sequence $< f_n >_{n \geq 1}$ converges uniformly to f on A. Prove that f is uniformly continuous on A.

7. Prove that $e^x > 7(x - 1)$ for all $x \geq 2$.

Five more questions on the back !!!

8. Let $f : [a, b] \mapsto \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b) = 0$ and that there exists $c \in (a, b)$ such that $f(c) > 0$. Prove there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) < 0 < f'(x_2)$.

9. Let $E \subseteq \mathbb{R}$ and $f, g, f_n, g_n : E \mapsto \mathbb{R}$ be functions for all $n \geq 1$ such that $< f_n >_{n\geq 1}$ converges uniformly to f on E and $< g_n >_{n\geq 1}$ converges uniformly to g on E. Suppose that f and g are bounded on E. Prove that $< f_n g_n >_{n\geq 1}$ converges uniformly to fg on E.

10. Evaluate $\displaystyle\lim_{n\mapsto+\infty} \int_0^1 \cos\left(\frac{x^2}{n}\right) dx$.

hint : Prove that $\cos\left(\frac{1}{n}\right) \leq \cos\left(\frac{x^2}{n}\right) \leq 1$ for all $x \in [0, 1]$. Use this to prove that the sequence $\left\langle \cos\left(\frac{x^2}{n}\right)\right\rangle_{n\geq 1}$ converges uniformly to 1 on $[0, 1]$.

11. Let $f : [0, 1] \mapsto \mathbb{R}$ be continuous. Prove that $\displaystyle\lim_{n\mapsto+\infty} \int_0^1 x^n f(x)\, dx = 0$.

12. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuous. Define $g : \mathbb{R} \mapsto \mathbb{R} : x \mapsto \displaystyle\int_{x^2}^{x^3} f(t + x)\, dt$. Calculate g'.

Mathematics Department Qualifying Exam **Fall 2004**
Subject : Analysis

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*Instructions : Solve **8** of the following problems :*

1. Consider the sequence $\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{2a_n + 3} \quad \text{if } n \geq 1 \end{cases}$

(a) Use induction on n to prove that $0 \leq a_n \leq 3$ for all $n \geq 1$.

(b) Prove that $< a_n >_{n\geq 1}$ is an increasing sequence.

(c) Deduce that $< a_n >_{n\geq 1}$ converges. Find $\lim\limits_{n \mapsto +\infty} a_n$.

2. Prove that $\sqrt[3]{2}$ is irrational.

3. Prove or give a counterexample : If $< F_n >_{n\geq 1}$ is a sequence of closed subsets of $\mathbb{R}$, then $\bigcup\limits_{n=1}^{+\infty} F_n$ is closed.

4. Define $g : \mathbb{R} \mapsto \mathbb{R} : g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases}$
Find (with proof) all the points at which g is continuous.

5. Let $X, Y \subset \mathbb{R}$ and $f : X \mapsto Y$ a function. For $B \subseteq Y$, we define $f^{-1}[B] = \{x \in X \,|\, f(x) \in B\}$.
Let I be an index set and $B_i \subseteq Y$ for all $i \in I$. Prove that $f^{-1}\left[\bigcap\limits_{i\in I} B_i\right] = \bigcap\limits_{i\in I} f^{-1}[B_i]$.

6. (a) State the Mean Value Theorem.
(b) Use the Mean Value Theorem to prove the following :

Let $f : (a,b) \mapsto \mathbb{R}$ be differentiable over (a,b). Suppose that $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant over (a,b).

7. If $A, B \subseteq \mathbb{R}$, we define $A + B = \{a + b \,|\, a \in A, b \in B\}$. Prove that $A + B$ is compact if A and B are compact.
hint : use the characterization of compact sets that involves sequences!

8. Let $f : D \mapsto \mathbb{R}$ be uniformly continuous over D and $< d_n >_{n\geq 1}$ a Cauchy sequence with $d_n \in D$ for all $n \geq 1$. Prove that $f(D) =< f(d_n) >_{n\geq 1}$ is a Cauchy sequence.

Four more questions on the back !!!

9. For $n \geq 1$, we define $f_n : [0,1] \mapsto \mathbb{R} : x \mapsto \dfrac{x^2 - x}{n^2}$

(a) Find the function $f : [0,1] \mapsto \mathbb{R}$ such that $< f_n >_{n \geq 1}$ converges pointwise to f.

(b) Is this convergence uniform? Prove your answer!

10. Consider the series $\displaystyle\sum_{n=1}^{+\infty} \frac{x}{n^2 + x^2}$. Prove that this series converges uniformly on $[0,1]$.

11. Let $f : [a,b] \mapsto \mathbb{R}$ be continuous over $[0,1]$. Suppose that $f(x) \geq 0$ for all $x \in [0,1]$. Prove that

$$\left[\int_0^1 f(x)\,dx\right]^2 \leq \int_0^1 f^2(x)\,dx$$

hint : For $n \geq 1$, put $\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Compare a Riemann sum with partition $\mathcal{P}_n$ for f to a Riemann sum with partition $\mathcal{P}_n$ for f^2.

12. Let $f, g : [0,1] \mapsto \mathbb{R}$ be continuous over $[0,1]$ such that $f(0) \leq g(0)$ and $f(1) \geq g(1)$. Prove that there exists $c \in [0,1]$ such that $f(c) = g(c)$.

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1. Prove that $\cos^2 x \geq 1 - x^2$ for all $x \in \mathbb{R}$.

Proof : Put $f(x) = \cos^2 x + x^2$. We need to prove that $f(x) \geq 1$ for all $x \in \mathbb{R}$. Since f is even (namely $f(x) = f(-x)$ for all $x \in \mathbb{R}$), it's enough to show that

$$f(x) \geq 1 \quad \text{for all } x \geq 0$$

We easily get that

$$f'(x) = -2\cos x \sin x + 2x \quad \text{and} \quad f''(x) = 2\sin^2 x - 2\cos^2 x + 2 = 4\sin^2 x$$

Hence

$$f''(x) \geq 0 \quad \text{for all } x \geq 0$$

So f' is increasing on $[0, +\infty)$. Hence

$$f'(x) \geq f'(0) = 0 \quad \text{for all } x \geq 0$$

So f is increasing on $[0, +\infty)$. Hence

$$f(x) \geq f(0) = 1 \quad \text{for all } x \geq 0$$

□

---

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq 2|x - y|^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Proof : Pick $a \in \mathbb{R}$. Then

$$|f(x) - f(a)| \leq 2|x - a|^2 \quad \text{for all } x \in \mathbb{R}$$

Hence

$$0 \leq \left| \frac{f(x) - f(a)}{x - a} \right| \leq 2|x - a| \quad \text{for all } x \neq a$$

Since $\lim_{x \to a} 0 = \lim_{x \to a} |x - a| = 0$, it follows from the Pinching Theorem that

$$\lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} \right| = 0$$

Hence

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0$$

So f is differentiable at a and $f'(a) = 0$. Since a was arbitrary, we get that f is differentiable over $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$. Hence f is constant. □

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3. Calculate $\lim_{n \to +\infty} \int_1^2 \frac{2 + nx^3}{3 + nx^2} \, dx$. Justify your answer!

Proof : Put $f_n : [1, 2] \to \mathbb{R} : x \to \frac{2 + nx^3}{3 + nx^2}$ for all $n \in \mathbb{N}$ and $f : [1, 2] \to \mathbb{R} : x \to x$. We prove that $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on $[0, 1]$. So we need to prove

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [1, 2] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \frac{8}{\epsilon}$. Pick $n \geq N$ and $x \in [1, 2]$. Note that $|2 - 3x| \leq 2 + 3|x| \leq 8$ and $|3 + nx^2| = 3 + nx^2 \geq n$. Hence

$$|f_n(x) - f(x)| = \left| \frac{2 + nx^3}{3 + nx^2} - x \right| = \left| \frac{2 - 3x}{3 + nx^2} \right| \leq \frac{2 + 3|x|}{3 + nx^2} \leq \frac{8}{n} \leq \frac{8}{N} < \epsilon$$

Since $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[1,2]$ and f_n is continuous (and hence Riemann Integrable) on $[1,2]$, we have that

$$\lim_{n\to\infty} \int_1^2 f_n(x)\,dx = \int_1^2 \lim_{n\to\infty} f_n(x)\,dx = \int_1^2 f(x)\,dx$$

Hence

$$\lim_{n\to+\infty} \int_1^2 \frac{2+nx^3}{3+nx^2}\,dx = \int_1^2 x\,dx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{3}{2} \qquad \square$$

---

4. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is f differentiable at $(0,0)$? Prove your answer!

<u>*Proof*</u> : Using the definition of partial derivatives, we get

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2+0^2}-0}{h} = \cdots = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^3}{0^2+h^2}-0}{h} = \cdots = 0$$

Next, we need to check if

$$\lim_{(x,y)\to(0,0)} \frac{f((0,0)+(x,y))-f(0,0)-\langle 1,0\rangle \cdot \langle x,y\rangle}{\|(x,y)\|} = 0$$

We easily get that for all $(x,y) \neq (0,0)$

$$\frac{f((0,0)+(x,y))-f(0,0)-\langle 1,0\rangle \cdot \langle x,y\rangle}{\|(x,y)\|} = \frac{\frac{x^3}{x^2+y^2}-x}{\sqrt{x^2+y^2}} = -\frac{xy^2}{(x^2+y^2)^{\frac{3}{2}}}$$

Let $k > 0$ and consider the half-line $y = kx$ where $x > 0$.
Then

$$\lim_{x\to 0^+} -\frac{x(kx)^2}{(x^2+(kx)^2)^{\frac{3}{2}}} = \lim_{x\to 0^+} -\frac{k^2x^3}{((1+k^2)x^2)^{\frac{3}{2}}} = -\frac{k^2}{(1+k^2)^{\frac{3}{2}}}$$

Since this limit clearly depends on k, we get that f is NOT differentiable at $(0,0)$. $\square$

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5. (a) State the ϵ-δ definition of continuity of a function of one variable :

$$f(x) \text{ is continuous at } x = a \iff \cdots$$

(b) Use your definition from (a) to prove that the function $f(x) = \dfrac{x}{x+1}$ is continuous at $x = -2$.

<u>*Proof*</u> : (a) $f(x)$ is continuous at $x = a$ iff

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in D : |x-a| < \delta \implies |f(x)-f(a)| < \epsilon$$

(b) Note that the domain of f is $\mathbb{R} \setminus \{2\}$ and $f(-2) = 2$. So we need to prove

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} \setminus \{2\} : |x+2| < \delta \implies \left|\frac{x}{x+1} - 2\right| < \epsilon$$

Pick $\epsilon > 0$. Let $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Pick $x \in \mathbb{R}\setminus\{2\}$ with $|x+2| < \delta$. In particular, $|x+2| < \frac{1}{2}$. Hence $-\frac{1}{2} < x+2 < \frac{1}{2}$. So $-\frac{3}{2} < x+1 < -\frac{1}{2}$ and $|x+1| > \frac{1}{2}$. Hence

$$\left|\frac{x}{x+1} - 2\right| = \left|\frac{-x-2}{x+1}\right| = \frac{|x+2|}{|x+1|} < \frac{\delta}{\frac{1}{2}} = 2\delta \leq \epsilon \qquad \square$$

---

6. Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above. For $\lambda \in \mathbb{R}$, we define $\lambda S = \{\lambda s \mid s \in S\}$. Prove that $\inf(-2S) = -2\sup S$.

Proof : Put $\alpha = \sup S$.
Pick $x \in -2S$. Then $x = -2s$ for some $s \in S$. Since α is an upper bound for S, we have that $s \leq \alpha$. Hence $x = -2s \geq -2\alpha$. Since x was arbitrary, we have that -2α is a lower bound for $-2S$.
Suppose that β is a lower bound for $-2S$. Pick $s \in S$. Then $-2s \in -2S$ and so $\beta \leq -2s$ since β is a lower bound for $-2S$. Hence $s \leq -\frac{1}{2}\beta$. Since s was arbitrary, we get that $-\frac{1}{2}\beta$ is an upper bound for S. But α is the smallest upper bound for S. Hence $\alpha \leq -\frac{1}{2}\beta$ and so $-2\alpha \geq \beta$.
Hence -2α is the greatest lower bound for $-2S$. So $-2\sup S = -2\alpha = \inf(-2S)$. $\square$

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7. Let $\langle x_n \rangle_{n\geq 0}$ be the sequence defined by

$$\begin{cases} x_{n+1} = n - x_n \text{ for all } n \geq 0 \\ x_0 = 0 \end{cases}$$

Prove that $x_{2k} = x_{2k+1}$ for all $k \geq 0$.

Proof : We prove this by induction on k.
Note that $x_1 = 0 - x_0 = 0 - 0 = 0 = x_0$ which proves the case '$k = 0$'.
So assume that $x_{2k} = x_{2k+1}$ for $k = 0, 1, \ldots, n-1$ for some $n \geq 1$. In particular, $x_{2n-2} = x_{2n-1}$. Then

$$x_{2n} = (2n-1) - x_{2n-1} = (2n-1) - [(2n-2) - x_{2n-2}] = 1 + x_{2n-2}$$

and

$$x_{2n+1} = (2n) - x_{2n} = 2n - [(2n-1) - x_{2n-1}] = 1 + x_{2n-1}$$

Since $x_{2n-2} = x_{2n-1}$, we get that $x_{2n} = x_{2n+1}$, proving the case '$k = n$'. $\square$

---

8. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b) = 0$. Prove that for all $\lambda \in \mathbb{R}$, there exists $c \in (a,b)$ such that $f'(c) = \lambda f(c)$.

Proof : Let $\lambda \in \mathbb{R}$. Put $g : [a,b] \to \mathbb{R} : x \to e^{-\lambda x} f(x)$. Then g is continuous on $[a,b]$, differentiable on (a,b) and $g(a) = g(b) = 0$. Hence it follows from the Mean Value Theorem that

$$\frac{g(b) - g(a)}{b - a} = g'(c) \quad \text{for some } c \in (a,b)$$

So $g'(c) = 0$. We easily get that $g'(x) = -\lambda e^{-\lambda x} f(x) + e^{-\lambda x} f'(x) = e^{-\lambda x}(-\lambda f(x) + f'(x))$. Since $e^{-\lambda x} \neq 0$ for all $x \in \mathbb{R}$, we have that $-\lambda f(c) + f'(c) = 0$. So $f'(c) = \lambda f(c)$. $\square$

---

9. Let $\langle a_n \rangle_{n\geq 0}$ be a sequence of real numbers. Put $b_n = a_n - a_{n+1}$ for all $n \geq 0$.

(a) Prove that the series $\sum_{n=0}^{+\infty} b_n$ converges if and only if the sequence $\langle a_n \rangle_{n\geq 0}$ converges.

(b) If the series $\sum_{n=0}^{+\infty} b_n$ converges, to what does it converge?

Proof : Put $s_n = \sum_{k=0}^{n} b_k$ for all $n \in \mathbb{N}$. We easily get that

$$s_n = \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k - a_{k+1}) = (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_n - a_{n+1}) = a_0 - a_{n+1}$$

By definition, we have that

$$\sum_{n=0}^{+\infty} b_n \text{ converges } \iff \langle s_n \rangle_{n\geq 0} \text{ converges } \iff \langle a_0 - a_{n+1} \rangle_{n\geq 0} \text{ converges}$$

Since a_0 is a constant, it follows that

$$\langle a_0 - a_{n+1} \rangle_{n\geq 0} \text{ converges } \iff \langle a_{n+1} \rangle_{n\geq 0} \text{ converges } \iff \langle a_n \rangle_{n\geq 0} \text{ converges}$$

Suppose that $\sum_{n=0}^{+\infty} b_n$ (and hence also $\langle a_n \rangle_{n\geq 0}$) converges. Then

$$\sum_{n=0}^{+\infty} b_n = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} (a_0 - a_{n+1}) = a_0 - \lim_{n\to+\infty} a_{n+1} = a_0 - \lim_{n\to+\infty} a_n \qquad \square$$

---

10. Prove that the series $\sum_{n=1}^{+\infty} \frac{n\,\sin(nx)}{x^n + ne^x}$ converges uniformly over $[\pi, +\infty)$.

Proof : Pick $n \geq 1$. Then for all $x \in [\pi, +\infty)$, we have that

$$|n \sin(nx)| = n|\sin(nx)| \leq n \quad \text{ and } \quad |x^n + ne^x| = x^n + ne^x \geq x^n \geq \pi^n$$

Hence

$$\left| \frac{n\,\sin(nx)}{x^n + ne^x} \right| \leq \frac{n}{\pi^n} := M_n \quad \text{ for all } x \geq \pi$$

Next, we use the Ratio Test to prove that the series $\sum_{n=1}^{+\infty} \frac{n}{\pi^n}$ converges. We easily get that

$$\lim_{n\to+\infty} \left| \frac{M_{n+1}}{M_n} \right| = \lim_{n\to+\infty} \frac{\frac{n+1}{\pi^{n+1}}}{\frac{n}{\pi^n}} = \lim_{n\to+\infty} \frac{n+1}{n\pi} = \frac{1}{\pi}$$

Since $\lim_{n\to+\infty} \left| \frac{M_{n+1}}{M_n} \right| < 1$, we get that the series $\sum_{n=1}^{+\infty} \frac{n}{\pi^n}$ converges. Hence it follows from the Weierstrass M-test that the series $\sum_{n=1}^{+\infty} \frac{n\,\sin(nx)}{x^n + ne^x}$ converges uniformly over $[\pi, +\infty)$. $\square$

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11. Let (X, d) be a metric space. Define $D : X \times X \to \mathbb{R} : (x, y) \to \frac{d(x, y)}{2^{d(x,y)}}$.
Is (X, D) a metric space? Prove your answer!

Proof : **NO** : the Triangle Inequality fails. Consider the regular distance on $\mathbb{R}$. Put $x = 0$, $y = 1$ and $z = 5$. Then

$$d(x,y) = |0-1| = 1 \quad , \quad d(x,z) = |0-5| = 5 \quad \text{and} \quad d(y,z) = |1-5| = 4$$

Hence

$$D(x,y) = \frac{1}{2} \quad , \quad D(x,z) = \frac{5}{2^5} = \frac{5}{32} \quad \text{and} \quad D(y,z) = \frac{4}{2^4} = \frac{1}{4}$$

But

$$D(x,y) = \frac{1}{2} \not\leq \frac{13}{32} = \frac{5}{32} + \frac{1}{4} = D(x,z) + D(z,y) \qquad \square$$

---

12. Let $a < b < c$ and $f : (a,c) \to \mathbb{R}$ a function such that f is continuous at $x = b$ and f is uniformly continuous on (a,b) and on $[b,c)$. Prove that f is uniformly continuous on (a,c).

Proof : Since f is uniformly continuous on (a,b) ,we have that $\lim_{x\to a^+} f(x)$ exists. Since f is uniformly continuous on $[b,c)$,we have that $\lim_{x\to c^-} f(x)$ exists. Since f is continuous on (a,b), on $[b,c)$ and at $x = b$, we have that f is continuous on (a,c). Since $\lim_{x\to a^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist, it follows that f is uniformly continuous on (a,c). $\square$

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1. Let n be a positive integer. Prove that

$$(1-x^2)^n \geq 1 - nx^2 \quad \text{for all } x \in [0,1]$$

Proof : Put $f : [0,1] \to \mathbb{R} : x \to (1-x^2)^n + nx^2$. We need to show that

$$f(x) \geq 1 \quad \text{for all } x \in [0,1]$$

We easily get that

$$f'(x) = n(1-x^2)^{n-1}(-2x) + 2nx = 2nx\left(1 - (1-x^2)^{n-1}\right) \quad \text{for all } x \in [0,1]$$

For all $x \in [0,1]$, we have that $0 \leq x^2 \leq 1$ and so $0 \leq 1 - x^2 \leq 1$; hence $0 \leq (1-x^2)^{n-1} \leq 1$ and $0 \leq 1 - (1-x^2)^{n-1}$. It follows that

$$f'(x) \geq 0 \quad \text{for all } x \in [0,1]$$

Hence f is increasing on $[0,1]$. So

$$f(x) \geq f(0) = 1 \text{ for all } x \in [0,1]$$

□

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2. Let $\langle x_n \rangle_{n \geq 0}$ be a sequence of real numbers such that $|x_{n+1} - x_n| \leq 2^{-n}$ for all $n \in \mathbb{N}$. Prove that $\langle x_n \rangle_{n \geq 1}$ converges.

Proof : We will prove that the sequence $\langle x_n \rangle_{n \geq 0}$ is a Cauchy sequence. So we need to show :

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : |x_m - x_n| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > 1 - \log_2(\epsilon)$. Then $2^{N-1} > 2^{-\log_2(\epsilon)} = \frac{1}{\epsilon}$ and so $\frac{1}{2^{N-1}} < \epsilon$. Pick $m, n \in \mathbb{N}$ with $m, n \geq N$. If $m = n$ then $|x_m - x_n| = 0 < \epsilon$. So we may assume that $m > n$. Then

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq 2^{-(m-1)} + 2^{-(m-2)} + \cdots + 2^{-n} \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots \\
&= \frac{1}{2^{n-1}} \\
&\leq \frac{1}{2^{N-1}} \\
&< \epsilon
\end{aligned}$$

So the sequence $\langle x_n \rangle_{n \geq 0}$ is a Cauchy sequence and hence converges. □

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3. This exercise is about uniform continuity :

(a) State the definition of uniform continuity :

a function f is uniformly continuous on a set $D \subseteq \mathbb{R}$ if ...

(b) Use this definition to prove that the function $f(x) = x^2$ is uniformly continuous on $(0, 1)$.

Proof : (a) f is uniformly continuous on a set $D \subseteq \mathbb{R}$ if and only if

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

(b) We need to prove :

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in (0, 1) : |x - y| < \delta \implies |x^2 - y^2| < \epsilon$$

Pick $\epsilon > 0$. Let $\delta = \frac{\epsilon}{2}$. Pick $x, y \in (0, 1)$ with $|x - y| < \delta$. Then

$$|x^2 - y^2| = |(x + y)(x - y)| \leq (|x| + |y|)|x - y| < (1 + 1)\delta = 2\delta = \epsilon$$

□

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4. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \to \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Calculate $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$.

(b) Is $f(x, y)$ differentiable at $(0, 0)$? Justify your answer!

Proof : (a) Using the definition of partial derivatives, we get

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{h \cdot 0}{\sqrt{h^2+0^2}} - 0}{h} = \cdots = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{0 \cdot h}{\sqrt{0^2+h^2}} - 0}{h} = \cdots = 0$$

Next, we need to check if

$$\lim_{(x,y) \to (0,0)} \frac{f((0, 0) + (x, y)) - f(0, 0) - \langle 0, 0 \rangle \cdot \langle x, y \rangle}{\|(x, y)\|} = 0$$

We easily get that for all $(x, y) \neq (0, 0)$

$$\frac{f((0, 0) + (x, y)) - f(0, 0) - \langle 0, 0 \rangle \cdot \langle x, y \rangle}{\|(x, y)\|} = \frac{\frac{xy}{\sqrt{x^2+y^2}} - 0}{\sqrt{x^2 + y^2}} = \frac{xy}{x^2 + y^2}$$

Let $k \in \mathbb{R}$ and consider the line $y = kx$.
Then

$$\lim_{x \to 0} \frac{x(kx)}{x^2 + (kx)^2} = \lim_{x \to 0} \frac{kx^2}{(1 + k^2)x^2} = \frac{k}{1 + k^2}$$

Since this limit clearly depends on k, we get that f is NOT differentiable at $(0, 0)$. □

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5. Suppose that the series of real numbers $\sum_{k=0}^{+\infty} a_k$ converges absolutely. Prove that the series $\sum_{k=0}^{+\infty} a_k^4$ converges.

Proof : Since the series $\sum_{k=0}^{+\infty} a_k$ converges, it follows from the Zero-Test that the sequence $\langle a_k \rangle_{k \geq 0}$ converges to zero.
In particular, we have :

$$\exists N \in \mathbb{N} : \forall k \geq N : |a_k - 0| < 1$$

Since $|a_k| < 1$ for all $k \geq N$, it follows that

$$a_k^4 = |a_k|^4 \leq |a_k| \quad \text{for all } k \geq N$$

Since the series $\sum_{k=0}^{+\infty} a_k$ converges absolutely, we have that the series $\sum_{k=0}^{+\infty} |a_k|$ converges. Since $a_k^4 \leq |a_k|$ for all $k \geq N$, we get that the series $\sum_{k=0}^{+\infty} a_k^4$ converges by the Comparison test. □

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6. Let $S \subseteq \mathbb{R}$ be nonempty and bounded below. Put

$$L = \{x \in \mathbb{R} \,|\, x \text{ is a lower bound for } S\}$$

Prove that $\inf(S) = \sup(L)$.

Proof : Put $\alpha = \inf(S)$ and $\beta = \sup(L)$. Since α is a lower bound for S, we have that $\alpha \in L$. But β is an upper bound for L. So $\alpha \leq \beta$.
Suppose that $\alpha < \beta$. Since β is the smallest upper bound for L and $\alpha < \beta$, we have that α is not an upper bound for L. Hence $\alpha < \gamma$ for some $\gamma \in L$. Since $\gamma \in L$, we have that γ is a lower bound for S. But $\alpha < \gamma$, a contradiction since α is the largest lower bound for S.
Hence $\alpha = \beta$ or $\inf(S) = \sup(L)$. □

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7. Consider the function $f_n : [1, +\infty) \to \mathbb{R} : x \to \dfrac{nx}{nx^2+1}$ for all $n \geq 1$.

(a) Find $\lim_{n \to +\infty} f_n$ and prove that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly on $[1, +\infty)$.

(b) Evaluate $\lim_{n \to +\infty} \int_1^5 f_n(x)\, dx$. Justify your answer!

Proof : (a) Put $f : [1, +\infty) \to \mathbb{R} : x \to \dfrac{1}{x}$. We prove that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on $[1, +\infty)$. So we need to show :

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [1, +\infty) : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \dfrac{1}{\epsilon}$. Pick $n \geq N$ and $x \in [1, +\infty)$. Then $x(nx^2+1) \geq 1 \cdot (n+1) = n+1 > n$. Hence

$$|f_n(x) - f(x)| = \left| \frac{nx}{nx^2+1} - \frac{1}{x} \right| = \left| \frac{-1}{x(nx^2+1)} \right| = \frac{1}{x(nx^2+1)} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

(b) Since the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on $[1, +\infty)$ (and hence also on $[1, 5]$) and f_n is continuous (and hence Riemann integrable) on $[1, 5]$, we have that

$$\lim_{n \to +\infty} \int_1^5 f_n(x)\, dx = \int_1^5 \lim_{n \to +\infty} f_n(x)\, dx = \int_1^5 f(x)\, dx$$

Hence

$$\lim_{n \to +\infty} \int_1^5 \frac{nx}{nx^2+1}\, dx = \int_{-1}^5 \frac{1}{x}\, dx = [\ln(x)]_1^5 = \ln(5)$$ □

---

8. Prove that every Cauchy sequence of real numbers is bounded WITHOUT using the fact that a Cauchy sequence of real numbers converges.

Proof : Let $\langle x_n \rangle_{n\geq 1}$ be a Cauchy sequence. So we know

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : |x_m - x_n| < \epsilon$$

In particular, we get

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |x_m - x_n| < 1$$

Pick $n \geq N$. Then $|x_n - x_N| < 1$. So $-1 < x_n - x_N < 1$. Hence

$$-1 - |x_N| \leq -1 + x_N < x_n < 1 + x_N \leq 1 + |x_N|$$

So

$$|x_n| < 1 + |x_N| \quad \text{for all } n \geq N$$

Put $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x_N|\}$. Then clearly $|x_n| \leq M$ for all $n \geq 1$. So the sequence $\langle x_n \rangle_{n\geq 1}$ is bounded (by M). □

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9. Prove that the series $\displaystyle\sum_{k=1}^{+\infty} \frac{x^3 - \cos(kx)}{k^2 + 3x^2}$ converges uniformly on $[-2, 1]$.

Proof : Pick $k \geq 1$. Then for all $x \in [-2, 1]$, we have that

$$|x^3 - \cos(kx)| \leq |x^3| + |\cos(kx)| \leq 8 + 1 = 9$$

and

$$|k^2 + 3x^2| = k^2 + 3x^2 \geq k^2$$

Hence

$$\left| \frac{x^3 - \cos(kx)}{k^2 + 3x^2} \right| \leq \frac{9}{k^2} \qquad \text{for all } x \in [-2, 1]$$

Since the series $\displaystyle\sum_{k=1}^{+\infty} \frac{9}{k^2}$ converges (it's a p-series with $p = 2$), it follows from the Weierstrass M-test that the series $\displaystyle\sum_{k=1}^{+\infty} \frac{x^3 - \cos(kx)}{k^2 + 3x^2}$ converges uniformly on $[-2, 1]$. □

---

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous over $\mathbb{R}$. Suppose that $f(0) = 0$. Prove that f is differentiable at $x = 0$ if and only if there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous over $\mathbb{R}$ and $f(x) = x\, g(x)$ for all $x \in \mathbb{R}$.

Proof : By definition of differentiability, we have that f is differentiable at $x = 0$ if and only if $\displaystyle\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$ exists. Since $f(0) = 0$, we have

$$f \text{ is differentiable at } x = 0 \iff \lim_{x\to 0} \frac{f(x)}{x} \text{ exists.}$$

Suppose first that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous over $\mathbb{R}$ and $f(x) = x\, g(x)$ for all $x \in \mathbb{R}$. Then

$$\frac{f(x)}{x} = g(x) \qquad \text{for all } x \neq 0$$

Hence

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} g(x) = g(0)$$

since g is continuous at $x = 0$. Hence $\lim_{x\to 0} \frac{f(x)}{x}$ exists. So f is differentiable at $x = 0$.
Suppose next that f is differentiable at $x = 0$. Then $\lim_{x\to 0} \frac{f(x)}{x}$ exists. Consider the function

$$g : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} \dfrac{f(x)}{x} & \text{if } x \neq 0 \\ \lim\limits_{x\to 0} \dfrac{f(x)}{x} & \text{if } x = 0 \end{cases}$$

Since f is continuous on $\mathbb{R}$, it follows that g is continuous on $\mathbb{R} \setminus \{0\}$. Also,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x} = g(0)$$

So g is also continuous at $x = 0$. Hence g is continuous on $\mathbb{R}$. Since $f(0) = 0 = 0 \cdot g(0)$, we get that $f(x) = x\, g(x)$ for all $x \in \mathbb{R}$. □

---

11. Suppose that $f : [a, b] \to \mathbb{Q}$ is continuous over $[a, b]$. Prove that f is constant over $[a, b]$.

Proof : Suppose that f is not constant on $[a, b]$. Then there exist $a \leq c < d \leq b$ with $f(c) \neq f(d)$. Let r be an irrational number between $f(c)$ and $f(d)$. By the Intermediate Value Theorem, there exists $x \in [c, d]$ with $f(x) = r$, a contradiction since $f([a, b]) \subseteq \mathbb{Q}$.
Hence f is constant on $[a, b]$. □

---

12. For $n \geq 1$, put

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Prove that the sequence $\langle a_n \rangle_{n\geq 1}$ converges and that $0 \leq \lim_{n\to+\infty} a_n \leq \frac{1}{2}$.

Proof : We easily see that $a_n > 0$ for all $n \geq 1$.
For $n \geq 1$, we get that

$$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}{2 \cdot 4 \cdot 6 \cdots (2(n+1))} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{2n+1}{2n+2} = a_n \cdot \frac{2n+1}{2n+2} < a_n$$

So the sequence $\langle a_n \rangle_{n\geq 1}$ is a decreasing seqeuence that is bounded below (by 0). Hence the sequence $\langle a_n \rangle_{n\geq 1}$ converges.
Since the sequence $\langle a_n \rangle_{n\geq 1}$ is decreasing and $a_1 = \frac{1}{2}$, we get that

$$0 \leq a_n \leq \frac{1}{2} \quad \text{for all } n \geq 1$$

Considering the limit as $n \to +\infty$, we find

$$0 \leq \lim_{n\to+\infty} a_n \leq \frac{1}{2}$$ □

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1. Put $f : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that $f\,'(0) = 1$.

Proof : Using the definition of differentiability, we get

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x + 2x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x\to 0} \left(1 + 2x \sin\left(\frac{1}{x}\right)\right) = 1 + \lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right)$$

Since $0 \leq |\sin(t)| \leq 1$ for all $t \in \mathbb{R}$, we have that

$$0 \leq \left|2x \sin\left(\frac{1}{x}\right)\right| = 2|x| \left|\sin\left(\frac{1}{x}\right)\right| \leq 2|x| \quad \text{for all } x \neq 0$$

Since $\lim_{x\to 0} 0 = \lim_{x\to 0} 2|x| = 0$, it follows from the Pinching Theorem that

$$\lim_{x\to 0} \left|2x \sin\left(\frac{1}{x}\right)\right| = 0$$

Hence

$$\lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) = 0$$

So $f'(0) = 1 + 0 = 1$. □

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2. (a) State the ϵ-δ definition of a limit of a function of two variables :

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \cdots$$

(b) What is $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$? Prove your answer using the definition from (a).

Proof : (a) $\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \forall \epsilon > 0 : \exists \delta > 0 : \forall (x,y) \in D : 0 < \|(x,y) - (a,b)\| < \delta \implies |f(x,y) - L| < \epsilon$

(b) We will prove that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$. So we have to show

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall (x,y) \in \mathbb{R}^2 : 0 < \|(x,y) - (0,0)\| < \delta \implies \left|\frac{x^2 y}{x^2 + y^2} - 0\right| < \epsilon$$

Pick $\epsilon > 0$. Put $\delta = \epsilon$. Pick $(x,y) \in \mathbb{R}^2$ with $0 < \|(x,y)\| < \delta$. Note that $\|(x,y)\| = \sqrt{x^2 + y^2}$, $|x| \leq \|(x,y)\|$ and $|y| \leq \|(x,y)\|$. Hence

$$\left|\frac{x^2 y}{x^2 + y^2} - 0\right| \leq \frac{\|(x,y)\|^2 \cdot \|(x,y)\|}{\|(x,y)\|^2} = \|(x,y)\| < \delta = \epsilon$$ □

---

3. Prove that the series $\sum_{n=1}^{+\infty} \frac{\sqrt{n}\sin(x^n)}{x^4+n^2}$ converges uniformly over $\mathbb{R}$.

Proof : Pick $n \geq 1$. Then for all $x \in \mathbb{R}$, we have that

$$|\sqrt{n}\sin(x^n)| = \sqrt{n}\,|\sin(x^n)| \leq \sqrt{n} \quad \text{and} \quad x^4+n^2 \geq n^2$$

Hence

$$\left|\frac{\sqrt{n}\sin(x^n)}{x^4+n^2}\right| \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}} \quad \text{for all } x \in \mathbb{R}$$

Since the series $\sum_{n=1}^{+\infty} \frac{1}{n^{\frac{3}{2}}}$ converges (it's a p-series with $p = \frac{3}{2}$), it follows from the Weierstrass M-test that the series $\sum_{n=1}^{+\infty} \frac{\sqrt{n}\sin(x^n)}{x^4+n^2}$ converges uniformly over $\mathbb{R}$. $\square$

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4. Let $f : \mathbb{R} \to (0, +\infty)$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

(a) Find $f(0)$.

(b) Prove that $f(x-y) = \frac{f(x)}{f(y)}$ for all $x, y \in \mathbb{R}$.

(c) Prove that f is continuous at 0 if and only if f is continuous over $\mathbb{R}$.

Proof : (a) Putting $x = y = 0$, we find that $f(0) = (f(0))^2$. So $f(0) \in \{0, 1\}$. Since $f(0) \in (0, +\infty)$, we have that $f(0) = 1$.

(b) Substituting $y = -x$ into the property for f, we find

$$1 = f(0) = f(x-x) = f(x)f(-x)$$

Since $f(x) \neq 0$, we get

$$f(-x) = \frac{1}{f(x)} \quad \text{for all } x \in \mathbb{R}$$

Substituting $y = -y$ into the property for f, we get

$$f(x-y) = f(x)f(-y) = f(x)\frac{1}{f(y)} = \frac{f(x)}{f(y)} \quad \text{for all } x, y \in \mathbb{R}$$

(c) If f is continuous over $\mathbb{R}$ then clearly f is continuous at 0. So suppose that f is continuous at 0. Pick $a \in \mathbb{R}$. We need to show that f is continuous at a. So we need to prove :

$$\forall \epsilon < 0 : \exists \delta > 0 : \forall x \in \mathbb{R} : |x-a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Pick $\epsilon > 0$. Since f is continuous at 0, we have (recall that $f(0) = 1$ and that $f(a) \neq 0$) :

$$\exists \delta >: \forall x \in \mathbb{R} : |x - 0| < \delta \implies |f(x) - 1| < \frac{\epsilon}{f(a)}$$

Pick $x \in \mathbb{R}$ with $|x-a| < \delta$. Then $|(x-a) - 0| < \delta$ and so

$$|f(x-a) - 1| < \frac{\epsilon}{f(a)}$$

By (b),

$$|f(x-a)-1| = \left|\frac{f(x)}{f(a)}-1\right| = \left|\frac{f(x)-f(a)}{f(a)}\right| = \frac{|f(x)-f(a)|}{f(a)} < \frac{\epsilon}{f(a)}$$

Hence $|f(x)-f(a)| < \epsilon$.
So f is continuous at a. Since a was arbitrary, we get that f is continuous over $\mathbb{R}$. □

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5. Let $f : (a, b) \to \mathbb{R}$ be twice differentiable over (a, b) and $x_1 < x_2 < x_3$ points in (a, b) with $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there exists a point $c \in (a, b)$ with $f''(c) > 0$.

Proof : Note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Hence it follows from the Mean Value Theorem that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(y_1) \quad \text{for some } y_1 \in (x_1, x_2)$$

Since $f(x_1) > f(x_2)$ and $x_1 < x_2$, we get that $f'(y_1) < 0$.
Similarly, f is continuous on $[x_2, x_3]$ and differentiable on (x_2, x_3). Hence it follows from the Mean Value Theorem that

$$\frac{f(x_3)-f(x_2)}{x_3-x_2} = f'(y_2) \quad \text{for some } y_2 \in (x_2, x_3)$$

Since $f(x_3) > f(x_2)$ and $x_2 < x_3$, we get that $f'(y_2) > 0$.
Note that $x_1 < y_1 < x_2 < y_2 < x_3$. Since f is twice differentiable over (a, b), we get that f' is differentiable over (a, b) and hence continuous on (a, b). So f' is continuous on $[y_1, y_2]$ and differentiable on (y_1, y_2). It follows from the Mean Value Theorem that

$$\frac{f'(y_2)-f'(y_1)}{y_2-y_1} = f''(c) \quad \text{for some } c \in (y_1, y_2)$$

Since $f'(y_2) > 0 > f'(y_1)$ and $y_1 < y_2$, we get that $f''(c) > 0$. □

---

6. Let $f : [a, b] \to \mathbb{R}$ be continuous and positive over $[a, b]$. Prove there exists a point $c \in [a, b]$ with $f(c) = \sqrt{f(a)f(b)}$.

Proof : We may assume that $f(a) \leq f(b)$. Then

$$f(a) \leq \sqrt{f(a)f(b)} \leq f(b)$$

Indeed, since f is positive, we get

$$f(a) \leq \sqrt{f(a)f(b)} \iff (f(a))^2 \leq f(a)f(b) \iff 0 \leq f(a)f(b) - (f(a))^2 = f(a)(f(b)-f(a))$$

Similarly, we get that $\sqrt{f(a)f(b)} \leq f(b)$.
It now follows from the Intermediate Value Theorem that $\sqrt{f(a)f(b)} = f(c)$ for some $c \in [a, b]$. □

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7. For all $n \geq 1$, let $f_n : [0, 1] \to \mathbb{R}$ be continuous over $[0, 1]$. Suppose that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to some function f on $[0, 1]$. Prove the following :

$$\forall \epsilon > 0 : \exists N \in \mathbb{N}, \exists \delta > 0 : \forall n \geq N, \forall x, y \in [0, 1] : |x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$$

Proof : Pick $\epsilon > 0$. Since the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on $[0, 1]$, we get that

$$\exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [0, 1] : |f_n(x) - f(x)| < \frac{\epsilon}{3} \qquad (*)$$

Since the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$ and f_n is continuous on $[0,1]$ for all $n \geq 1$, we get that f is continuous on $[0,1]$. But $[0,1]$ is compact. Hence f is uniformly continuous on $[0,1]$. So we get

$$\exists \delta > 0 : \forall x, y \in [0,1] : |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{3} \qquad (**)$$

Pick $n \geq N$ and $x, y \in [0,1]$ with $|x-y| < \delta$. Using (*) and (**), we get

$$\begin{aligned} |f_n(x) - f_n(y)| &= |(f_n(x) - f(x)) + (f(x) - f(y)) + (f(y) - f_n(y))| \\ &\leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f_n(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

□

8.(a) For all $n \geq 1$, put $f_n : [0,1] \to \mathbb{R} : x \to \dfrac{n\cos(x)}{n + e^x}$. Prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly on $[0,1]$.

(b) Calculate $\displaystyle\lim_{n\to+\infty} \int_0^1 \frac{n\cos(x)}{n+e^x}\,dx$. Justify your answer!

Proof : (a) Put $f : [0,1] \to \mathbb{R} : x \to \cos(x)$. We prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$. So we need to show :

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [0,1] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \dfrac{e}{\epsilon}$. Pick $n \geq N$ and $x \in [0,1]$. Then $|e^x \cos(x)| = e^x |\cos(x)| \leq e^x \leq e$ and $n + e^x > n$. Hence

$$|f_n(x) - f(x)| = \left|\frac{n\cos(x)}{n+e^x} - \cos(x)\right| = \left|\frac{-e^x\cos(x)}{n+e^x}\right| = \frac{e^x|\cos(x)|}{n+e^x} < \frac{e}{n} \leq \frac{e}{N} < \epsilon$$

(b) Since the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$ and f_n is continuous (and hence Riemann integrable) on $[0,1]$, we have that

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to+\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

Hence

$$\lim_{n\to+\infty} \int_0^1 \frac{n\cos(x)}{n+e^x}\,dx = \int_0^1 \cos(x)\,dx = [\sin(x)]_0^1 = \sin(1)$$

□

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9. (a) State the definition of a Cauchy sequence.
(b) Suppose that $0 < a < 1$ and $\langle x_n \rangle_{n\geq 1}$ is a sequence of real numbers with $|x_{n+1} - x_n| < a^n$ for all $n \geq 1$. Prove that the sequence $\langle x_n \rangle_{n\geq 1}$ converges.

Proof : (a) The sequence $\langle x_n \rangle_{n\geq 1}$ is a Cauchy sequence if and only if

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : |x_m - x_n| < \epsilon$$

(b) We will prove that the sequence $\langle x_n \rangle_{n\geq 1}$ is a Cauchy sequence. Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \log_a(\epsilon(1-a))$. Then $a^N < a^{\log_a(\epsilon(1-a))} = \epsilon(1-a)$ and so $\dfrac{a^N}{1-a} < \epsilon$. Pick $m, n \in \mathbb{N}$ with $m, n \geq N$. If $m = n$ then $|x_m - x_n| = 0 < \epsilon$.

So we may assume that $m > n$. Then

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&< a^{m-1} + a^{m-2} + \cdots + a^n \\
&< a^n + a^{n+1} + a^{n+2} + \cdots \\
&= \frac{a^n}{1-a} \\
&\leq \frac{a^N}{1-a} \\
&< \epsilon
\end{aligned}$$

So the sequence $\langle x_n \rangle_{n\geq 1}$ is a Cauchy sequence and hence converges. □

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10. Let (M, d) be a metric space.

(a) Let $A \subseteq M$. State the definition of an open set. So

$$A \subseteq M \text{ is open } \iff \cdots$$

(b) Let $a \in M$ and $r > 0$. Prove that the set $B(a, r) := \{x \in M \,|\, d(a, x) < r\}$ is open.

Proof : (a) $A \subseteq M$ is open if and only if

$$\forall a \in A : \exists \delta > 0 : \forall x \in M : d(a, x) < \delta \implies x \in A$$

(b) We need to prove : $\forall y \in B(a, r) : \exists \delta > 0 : \forall x \in M : d(x, y) < \delta \implies x \in B(a, r)$.
Pick $y \in B(a, r)$. Then $d(a, y) < r$ and so $\delta := r - d(a, y) > 0$. Pick $x \in M$ with $d(x, y) < \delta$. it follows from the Triangle Inequality that

$$d(a, x) \leq d(a, y) + d(y, x) < d(a, y) + \delta = d(a, y) + r - d(a, y) = r$$

Hence $x \in B(a, r)$. □

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11. Let $A \subseteq \mathbb{R}$ be compact and $\lambda > 0$. We define $\lambda A := \{\lambda x \,|\, x \in A\}$. Prove that λA is compact.

Proof : Recall that a set D is compact if and only if every sequence in D has a subsequence that converges to some element in D.
Let $\langle y_n \rangle_{n\geq 1}$ be a sequence in λA. Then for all $n \geq 1$, we have that $y_n = \lambda x_n$ for some $x_n \in A$. Since A is compact, we have that the sequence $\langle x_n \rangle_{n\geq 1}$ has a subsequence $\langle x_{n_k} \rangle_{k\geq 1}$ that converges to some element $a \in A$. Since $y_{n_k} = \lambda x_{n_k}$ for all $k \geq 1$, we get that $\langle y_{n_k} \rangle_{k\geq 1}$ is a subsequence of $\langle y_n \rangle_{n\geq 1}$ that converges to λa, which is an element of λA.
Hence λA is compact. □

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12. Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded.

(a) Prove there exists a sequence $\langle s_n \rangle_{n\geq 1}$ in S that converges to $\sup S$.

(b) Is $\sup S \in S$?

Proof : (a) Put $\alpha = \sup S$. Since α is an upper bound for S, we have

$$\forall s \in S : s \leq \alpha \qquad (*)$$

Since α is the smallest upper bound for S, we get

$$\forall \epsilon > 0 : \exists s \in S : \alpha - \epsilon < s \qquad (**)$$

It follows from (**) that

$$\forall n \geq 1 : \exists s_n \in S : \alpha - \frac{1}{n} < s_n$$

Combining with (*), we find

$$\alpha - \frac{1}{n} < s_n \leq \alpha \quad \text{for all } n \geq 1$$

Since $\lim_{n \to +\infty} \left(\alpha - \frac{1}{n} \right) = \lim_{n \to +\infty} \alpha = \alpha$, it follows from the Pinching Theorem that $\lim_{n \to +\infty} s_n = \alpha$.
So $\langle s_n \rangle_{n \geq 1}$ is a sequence in S that converges to $\alpha = \sup(S)$.

(b) **NO** : Put $S = (0, 1)$. Then $\sup S = 1 \notin S$. □

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1. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n\cos(nx)}{e^x + n^3}$ converges uniformly over $\mathbb{R}$.

Proof : Pick $n \geq 1$. Then for all $x \in \mathbb{R}$, we have that $|n\cos(nx)| = n|\cos(nx)| \leq n$ and $|e^x + n^3| = e^x + n^3 \geq n^3$. Hence

$$\left|\frac{n\cos(nx)}{e^x + n^3}\right| \leq \frac{n}{n^3} = \frac{1}{n^2} \quad \text{for all } x \in \mathbb{R}$$

Since the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges (it's p-series with $p = 2$) it follows from the Weierstrass M-test that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n\cos(nx)}{e^x + n^3}$ converges uniformly over $\mathbb{R}$. □

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2. (a) Let I be an interval, $f : I \to \mathbb{R}$ a function and $c \in I$. State the ϵ-δ definition :

f is continuous at c if and only if ...

(b) Use the definition you gave in (a) to prove that $f(x) = \dfrac{x}{x^2 + 1}$ is continuous at $x = 1$.

Proof : (a) f is continuous at c if and only if

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in I : |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$

(b) Note that the domain of f is $\mathbb{R}$ and $f(1) = \dfrac{1}{2}$. So we need to prove

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} : \left|\frac{x}{x^2 + 1} - \frac{1}{2}\right| < \epsilon$$

Pick $\epsilon > 0$. Put $\delta = \min\{1, 2\epsilon\}$. Pick $x \in \mathbb{R}$ with $|x - 1| < \delta$. Note that $x^2 + 1 \geq 1$ and $\delta^2 \leq \delta$ since $0 < \delta \leq 1$. Hence

$$\left|\frac{x}{x^2 + 1} - \frac{1}{2}\right| = \left|\frac{2x - x^2 - 1}{2(x^2 + 1)}\right| = \frac{|x - 1|^2}{2(x^2 + 1)} < \frac{\delta^2}{2 \cdot 1} \leq \frac{\delta}{2} \leq \epsilon$$ □

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3. Prove that the sequence $\displaystyle\left\langle \frac{n\ \sin(n)}{n + 1} \right\rangle_{n \geq 1}$ has a convergent subsequence.

Proof : Note that

$$\left|\frac{n\sin(n)}{n + 1}\right| = \frac{n|\sin(n)|}{n + 1} \leq \frac{n}{n + 1} < 1 \quad \text{for all } n \geq 1$$

Hence the sequence $\displaystyle\left\langle \frac{n\ \sin(n)}{n + 1} \right\rangle_{n \geq 1}$ is bounded. By the Bolzano-Weierstrass Theorem, the sequence $\displaystyle\left\langle \frac{n\ \sin(n)}{n + 1} \right\rangle_{n \geq 1}$ has a convergent subsequence. □

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4. Prove that $1 + 2x\ln(x) \leq x^2$ for all $x \geq 1$.

Proof : Put $f(x) = x^2 - 2x\ln(x)$. We need to prove that

$$f(x) \geq 1 \quad \text{for all } x \geq 1$$

We easily get that

$$f'(x) = 2x - 2\ln(x) - 2 \quad \text{and} \quad f''(x) = 2 - \frac{2}{x} = \frac{2(x-1)}{x} \quad \text{for all } x \geq 1$$

Hence

$$f''(x) \geq 0 \quad \text{for all } x \geq 1$$

So f' is increasing on $[1, +\infty)$. Hence

$$f'(x) \geq f'(1) = 0 \quad \text{for all } x \geq 1$$

So f is increasing on $[1, +\infty)$. Hence

$$f(x) \geq f(1) = 1 \quad \text{for all } x \geq 1$$

□

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5. (a) Calculate $\lim_{x\to 0^+} \sqrt{x}\ln(x)$.

(b) Prove that $f(x) = \sqrt{x}\ln(x)$ is uniformly continuous over $(0,1)$ (do not use the definition of uniform continuity).

Proof : (a) We easily get that

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = 0 \cdot (-\infty)$$

Hence we rewrite the limit and use l'Hospital's Rule :

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}}} \stackrel{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim_{x\to 0^+} \left(-2x^{\frac{1}{2}}\right) = 0$$

(b) It follows from (a) that $\lim_{x\to 0^+} f(x)$ exists. Clearly, $\lim_{x\to 1^-} f(x)$ exists since f is continuous at $x = 1$. Hence f is uniformly continuous on $(0,1)$. □

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6. Calculate $\lim_{n\to+\infty} n \int_0^{\frac{1}{n}} \cos(x^2)\, dx$. Justify your answer!

Proof : Pick $n \geq 1$. By the Intermediate Value Theorem for integrals, we get that

$$\int_0^{\frac{1}{n}} \cos(x^2)\, dx = \left(\frac{1}{n} - 0\right)\cos(c_n^2) \quad \text{for some } c_n \in \left(0, \frac{1}{n}\right)$$

Hence

$$n\int_0^{\frac{1}{n}} \cos(x^2)\, dx = \cos(c_n^2)$$

Since $0 < c_n < \frac{1}{n}$ for all $n \geq 1$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$, it follows from the Pinching Theorem that $\langle c_n \rangle_{n\geq 1} \to 0$. Since the function $\cos(x^2)$ is continuous at $x = 0$ and $\langle c_n \rangle_{n\geq 1} \to 0$, we get that

$$\lim_{n\to+\infty} n\int_0^{\frac{1}{n}} \cos(x^2)\, dx = \lim_{n\to+\infty} \cos(c_n^2) = \cos\left(\left(\lim_{n\to+\infty} c_n\right)^2\right) = \cos(0^2) = 1$$

□

---

7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable over $\mathbb{R}$. Prove that f is continuous over $\mathbb{R}$.

Proof : Pick $a \in \mathbb{R}$. Since f is differentiable at $x = a$, we get that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Hence

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\left((x-a)\frac{f(x)-f(a)}{x-a}\right) = \left(\lim_{x\to a}(x-a)\right)\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) = 0\cdot f'(a) = 0$$

So $\lim_{x\to a} f(x) = f(a)$. Hence f is continuous at $x = a$. Since a was arbitrary, we get that f is continuous over $\mathbb{R}$. □

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8. For $n \geq 1$, put

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

Prove that the sequence $\langle x_n\rangle_{n\geq 1}$ converges.

<u>*Proof*</u> : Pick $n \geq 1$. Then

$$\begin{aligned} x_n \leq x_{n+1} \quad &\iff \quad \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n} \leq \frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2(n+1)-1}+\frac{1}{2(n+1)} \\ &\iff \quad \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n} \leq \frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+1}+\frac{1}{2n+2} \\ &\iff \quad \frac{1}{n+1} \leq \frac{1}{2n+1}+\frac{1}{2n+2} \\ &\iff \quad \frac{1}{2n+2} \leq \frac{1}{2n+1} \\ &\iff \quad 2n+1 \leq 2n+2 \end{aligned}$$

So the sequence $\langle x_n\rangle_{n\geq 1}$ is increasing.
Pick $n \geq 1$. Note that the formula for x_n contains n terms :

$$x_n = \sum_{k=1}^{n}\frac{1}{n+k}$$

Since $n+k \geq n$ for $k = 0, 1, \ldots, n$, we easily get that

$$x_n = \sum_{k=1}^{n}\frac{1}{n+k} \leq \sum_{k=1}^{n}\frac{1}{n} = 1$$

Hence the sequence $\langle x_n\rangle_{n\geq 1}$ is increasing and bounded above (by 1). So the sequence $\langle x_n\rangle_{n\geq 1}$ converges. □

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9. Let (M, d) be a metric space. We define $d^2 : M \times M \to \mathbb{R} : (x, y) \to (d(x, y))^2$. Is (M, d^2) a metric space? Prove your answer!

<u>*Proof*</u> : **NO** : the Triangle Inequality fails. Consider $\mathbb{R}$ with the regular distance. Put $x = 0$, $y = 2$ and $z = 1$. Then

$$d(x,y) = |0-2| = 2 \;\;,\;\; d(x,z) = |0-1| = 1 \;\text{ and }\; d(y,z) = |2-1| = 1$$

Hence

$$d^2(x,y) = 4 \;\;, d^2(x,z) = 1 \;\text{ and }\; d^2(y,z) = 1$$

But

$$d^2(x,y) = 4 \not\leq 2 = 1+1 = d^2(x,z) + d^2(y,z)$$

□

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10. Prove that $f(x) = \dfrac{x}{x+1}$ is uniformly continuous over $[0, +\infty)$ using the definition of uniform continuity.

Proof : We need to prove

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in [0, +\infty) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Pick $\epsilon > 0$. Put $\delta = \epsilon$. Pick $x, y \in [0, +\infty)$ with $|x - y| < \delta$. Note that $x + 1 \geq 1$ and $y + 1 \geq 1$. Hence

$$|f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \left| \frac{x-y}{(x+1)(y+1)} \right| = \frac{|x-y|}{(x+1)(y+1)} < \frac{\delta}{1 \cdot 1} = \delta = \epsilon \qquad \square$$

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11. Let $A = \mathbb{Q} \cap [0, 1]$. Is A compact? Prove your answer!

Proof : Recall that a set A is compact if and only if every sequence in A has a subsequence that converges to some element in A.
We can easily find a sequence $\langle x_n \rangle_{n \geq 1}$ in A that converges to some irrational number r (e.g. let x_n be the rational number we get by considering the first n digits of the decimal expansion of $\frac{1}{\sqrt{2}}$; so $x_n = \left[\frac{10^n}{\sqrt{2}}\right]$). Any subsequence of $\langle x_n \rangle_{n \geq 1}$ will converge to r. Hence A is not compact. $\square$

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12. Calculate $\lim_{n \to +\infty} \int_0^1 \frac{nx}{n + x^3} \, dx$. Justify your answer!

Proof : Put $f_n : [0, 1] \to \mathbb{R} : x \to \dfrac{nx}{n + x^3}$ for all $n \geq 1$ and $f : [0, 1] \to \mathbb{R} : x \to x$. We prove that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f over $[0, 1]$. So we need to prove

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [0, 1] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Pick $n \geq N$ and $x \in [0, 1]$. Note that $x^4 \leq 1$ and $n + x^3 \geq n$. Hence

$$|f_n(x) - f(x)| = \left| \frac{nx}{n + x^3} - x \right| = \left| \frac{-x^4}{n + x^3} \right| = \frac{x^4}{n + x^3} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

So the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f over $[0, 1]$. Since f_n is continuous (and hence Riemann integrable) on $[0, 1]$, we get

$$\lim_{n \to +\infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to +\infty} f_n(x) \, dx = \int_0^1 f(x) \, dx$$

So

$$\lim_{n \to +\infty} \int_0^1 \frac{nx}{n + x^3} \, dx = \int_0^1 x \, dx = \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} \qquad \square$$

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1. Let X and Y be sets and $f : X \to Y$ a function. Recall that $f(A) = \{f(a) \,|\, a \in A\}$ for all $A \subseteq X$. Prove that f is one-to-one on X if and only if $f(A \setminus B) = f(A) \setminus f(B)$ for all $A, B \subseteq X$.

Proof : Suppose first that f is one-to-one. Let $A, B \subseteq X$. Pick $y \in f(A \setminus B)$. Then $y = f(x)$ for some $x \in A \setminus B$. So $y = f(x) \in f(A)$. Suppose $y \in f(B)$. Then $y = f(x')$ for some $x' \in B$. Since f is one-to-one, we get that $x = x'$, a contradiction since $x \in A \setminus B$ and $x' \in B$. Hence $y \notin f(B)$ and so $y \in f(A) \setminus f(B)$. Thus $f(A \setminus B) \subseteq f(A) \setminus f(B)$. Let $y \in f(A) \setminus f(B)$. Then $y = f(x)$ for some $x \in A$ since $y \in f(A)$. But $y \notin f(B)$ and so $x \notin B$. Hence $x \in A \setminus B$ and so $y = f(x) \in f(A \setminus B)$. Hence $f(A) \setminus f(B) \subseteq f(A \setminus B)$ and so $f(A \setminus B) = f(A) \setminus f(B)$.

Suppose next that $f(A \setminus B) = f(A) \setminus f(B)$ for all $A, B \subseteq X$. Let $x, x' \in X$ with $f(x) = f(x')$. Suppose $x \neq x'$. Put $A = \{x\}$ and $B = \{x'\}$. Then

$$\{f(x)\} = f(\{x\}) = f(\{x\} \setminus \{x'\}) = f(A \setminus B) = f(A) \setminus f(B) = f(\{x\}) \setminus f(\{x'\}) = \{f(x)\} \setminus \{f(x')\} = \emptyset$$

a contradiction. Hence $x = x'$. So f is one-to-one. □

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2. Prove that $\cos(x) \geq 1 - \dfrac{x^2}{2}$ for all $x \geq 0$.

Proof : Put $f(x) = \cos(x) + \dfrac{x^2}{2}$. We need to prove that $f(x) \geq 1$ for all $x \geq 0$. We easily get that

$$f'(x) = -\sin(x) + x \quad \text{and} \quad f''(x) = -\cos(x) + 1 \qquad \text{for all } x \geq 0$$

Since $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$, we see that $f''(x) \geq 0$ for all $x \geq 0$. Hence f' is increasing on $[0, +\infty)$. So

$$f'(x) \geq f'(0) = 0 \quad \text{for all } x \geq 0$$

Hence f is increasing on $[0, +\infty)$. So

$$f(x) \geq f(0) = 1 \text{ for all } x \geq 0.$$

□

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3. (a) Let $\langle \vec{a}_n \rangle_{n \geq 1}$ be a sequence in the k-dimensional Euclidean space $\mathbb{R}^k$ and $\vec{a} \in \mathbb{R}^k$. State the ε-N-definition : $\langle \vec{a}_n \rangle_{n \geq 1}$ converges to $\vec{a}$ if . . .

(b) Let $\langle \vec{a}_n \rangle_{n \geq 1}$ (respectively $\langle \vec{b}_n \rangle_{n \geq 1}$) be a sequence in $\mathbb{R}^k$ that converges to $\vec{a} \in \mathbb{R}^k$ (respectively $\vec{b} \in \mathbb{R}^k$). To which element of $\mathbb{R}^k$ does the sequence $\langle 2\vec{a}_n - 3\vec{b}_n \rangle_{n \geq 1}$ converge? Prove your answer using the definition you stated in (a).

Proof : (a) The sequence $\langle \vec{a}_n \rangle_{n \geq 1}$ converges to $\vec{a}$ if and only if and only if

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \|\vec{a}_n - \vec{a}\| < \epsilon$$

(b) We prove that the sequence $\langle 2\vec{a}_n - 3\vec{b}_n \rangle_{n \geq 1}$ converges to $2\vec{a} - 3\vec{b}$. Pick $\epsilon > 0$. Since the sequence $\langle \vec{a}_n \rangle_{n \geq 1}$ converges to $\vec{a}$ and the sequence $\langle \vec{b}_n \rangle_{n \geq 1}$ converges to $\vec{b}$, we have

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1 : \|\vec{a}_n - \vec{a}\| < \frac{\epsilon}{5}$$

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2 : \|\vec{b}_n - \vec{b}\| < \frac{\epsilon}{5}$$

Put $N = \max\{N_1, N_2\}$. Pick $n \geq 5$. Then it follows from the Triangle Inequality that

$$\|(2\vec{a}_n - 3\vec{b}_n) - (2\vec{a} - 3\vec{b})\| = \|(2(\vec{a}_n - \vec{a}) - 3(\vec{b}_n - \vec{b})\| \leq 2\|\vec{a}_n - \vec{a}\| + 3\|\vec{b}_n - \vec{b}\| < 2 \cdot \frac{\epsilon}{5} + 3 \cdot \frac{\epsilon}{5} = \epsilon \qquad \square$$

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4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose that f is continuous at $x = 0$. Prove that f is continuous over $\mathbb{R}$.

Proof : Putting $x = y = 0$, we get that $f(0) = f(0) + f(0)$ and so $f(0) = 0$. Substituting $y = -x$, we get that $0 = f(0) = f(x - x) = f(x) + f(-x)$ and so $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Substituting $y = -y$, we get that $f(x - y) = f(x) + f(-y) = f(x) - f(y)$ for all $x, y \in \mathbb{R}$.
Pick $a \in \mathbb{R}$. We need to prove that f is continuous at $x = a$. So we need to show

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Pick $\epsilon > 0$. Since f is continuous at $x = 0$, we have (recall that $f(0) = 0$) :

$$\exists \delta > 0 : \forall x \in \mathbb{R} : |x - 0| < \delta \implies |f(x)| < \epsilon$$

Pick $x \in \mathbb{R}$ with $|x - a| < \delta$. Then $|(x - a) - 0| < \delta$ and so

$$|f(x) - f(a)| = |f(x - a)| < \epsilon$$

Hence f is continuous at $x = a$. Since a was arbitrary, we get that f is continuous over $\mathbb{R}$. $\square$

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5. Let $f : (a, b) \to \mathbb{R}$ be differentiable over (a, b) such that f' is bounded on (a, b). Prove that f is uniformly continuous over (a, b).

Proof : We need to show that

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in (a, b) : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Pick $\epsilon > 0$. Since f' is bounded on (a, b), we have

$$\exists M > 0 : \forall x \in (a, b) : |f'(x)| \leq M$$

Put $\delta = \dfrac{\epsilon}{M}$. Pick $x, y \in (a, b)$ with $|x - y| < \delta$. If $x = y$ then $|f(x) - f(y)| = 0 < \epsilon$. So we may assume that $x \neq y$, say $x > y$. Note that f is continuous on $[y, x]$ and differentiable on (y, x). Hence it follows from the Mean Value Theorem that

$$\frac{f(x) - f(y)}{x - y} = f'(c) \quad \text{for some } c \in (y, x)$$

Hence

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y| < M\delta = \epsilon \qquad \square$$

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6. Let $f : [1, 2] \to [0, 4]$ be a continuous function such that $f(1) = 0$ and $f(2) = 3$. Prove that there exists $c \in [1, 2]$ such that $f(c) = c$.

Consider the function $g : [1, 2] \to \mathbb{R} : x \to f(x) - x$. Then g is continuous on $[1, 2]$, $g(1) = f(1) - 1 = 0 - 1 = -1 < 0$ and $g(2) = f(2) - 2 = 3 - 2 = 1 > 0$. By the Intermediate Value Theorem, we get that $g(c) = 0$ for some $c \in [1, 2]$. Hence $f(c) = c$. $\square$

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7. Evaluate $\lim_{n\to+\infty}\int_0^1 \frac{x}{x^2+ne^x}\,dx$. Justify your answer!

<u>*Proof*</u> : For $n \geq 1$, put $f_n : [0,1] \to \mathbb{R} : x \to \frac{x}{x^2+ne^x}$ and $f : [0,1] \to \mathbb{R} : x \to 0$. We will prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$. So we need to show

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [0,1] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Pick $n \geq N$ and $x \in [0,1]$. Then $|x| = x \leq 1$ and $|x^2+ne^x| = x^2+ne^x \geq ne^x \geq n$. Hence

$$|f_n(x) - f(x)| = \left|\frac{x}{x^2+ne^x} - 0\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Since the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$ and f_n is continuous (and hence Riemann integrable) on $[0,1]$, we have that

$$\lim_{n\to+\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to+\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

So

$$\lim_{n\to+\infty}\int_0^1 \frac{x}{x^2+ne^x}\,dx = \int_0^1 0\,dx = 0 \qquad \square$$

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8. Prove that the series $\sum_{n=1}^{+\infty}\frac{1}{n}e^{-nx}$ converges uniformly over $[1,+\infty)$ to some function f. Find a closed form for $f'(x)$.

<u>*Proof*</u> : For all $n \geq 1$ and all $x \in [1,+\infty)$, we have that

$$\left|\frac{1}{n}e^{-nx}\right| = \frac{1}{n}e^{-nx} \leq e^{-nx} \leq e^{-n}$$

Since the series $\sum_{n=1}^{+\infty} e^{-n}$ converges (it's a geometric series with ratio e^{-1}), it follows from the Weierstrass M-test that the series $\sum_{n=1}^{+\infty}\frac{1}{n}e^{-nx}$ converges uniformly over $[1,+\infty)$ to some function f.

Put $f_n : [1,+\infty) \to \mathbb{R} : x \to \frac{1}{n}e^{-nx}$ for all $n \geq 1$. Note that $f_n'(x) = -e^{-nx}$ for all $n \geq 1$. Moreover, for all $n \geq 1$ and all $x \in [1,+\infty)$, we have that

$$|-e^{-nx}| = e^{-nx} \leq e^{-n}$$

Since the series $\sum_{n=1}^{+\infty} e^{-n}$ converges (it's a geometric series with ratio e^{-1}), it follows from the Weierstrass M-test that the series $\sum_{n=1}^{+\infty}\left(-e^{-nx}\right)$ converges uniformly over $[1,+\infty)$ to some function g. Since the series $\sum_{n=1}^{+\infty} f_n$ converges uniformly on $[1,+\infty)$ to f and the series $\sum_{n=1}^{+\infty} f_n'$ converges uniformly on $[1,+\infty)$ to g, we have that $f' = g$ on $[1,+\infty)$. Hence

$$f'(x) = g(x) = \sum_{n=1}^{+\infty} f_n'(x) = \sum_{n=1}^{+\infty}\left(-e^{-nx}\right) = -\frac{e^{-x}}{1-e^{-x}} = \frac{1}{1-e^x} \quad \text{for all } x \geq 1 \qquad \square$$

---

9. Let (M, d) be a metric space. For $x, y \in M$ we define $e(x, y) = \min\{1, d(x, y)\}$. Prove that (M, e) is a metric space.

Proof : Pick $x, y \in M$. Since $1 \geq 0$ and $d(x, y) \geq 0$, we get that $e(x, y) = \min\{1, d(x, y)\} \geq 0$.
Also, $e(x, y) = 0 \iff \min\{1, d(x, y)\} = 0 \iff d(x, y) = 0 \iff x = y$.
Since $d(x, y) = d(y, x)$, we have that $e(x, y) = \min\{1, d(x, y)\} = \min\{1, d(y, x)\} = e(y, x)$.
Pick $x, y, z \in M$. We need to prove the Triangle Inequality :

$$e(x, y) \leq e(x, z) + e(z, y)$$

If $d(x, z) \geq 1$ or $d(z, y) \geq 1$ then $e(x, z) = \min\{1, d(x, z)\} = 1$ or $e(z, y) = \min\{1, d(z, y)\} = 1$; hence

$$e(x, y) = \min\{1, d(x, y)\} \leq 1 \leq e(x, z) + e(z, y)$$

So we may assume that $d(x, z) < 1$ and $d(z, y) < 1$. Then by the Triangle Inequality, we get

$$e(x, y) = \min\{1, d(x, y)\} \leq d(x, y) \leq d(x, z) + d(z, y) = \min\{1, d(x, z)\} + \min\{1, d(z, y)\} = e(x, z) + e(z, y)$$ □

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10. Let $\langle a_n \rangle_{n \geq 1}$ be a sequence of real numbers that converges to $\alpha \in \mathbb{R}$. Suppose that $\beta \in \mathbb{R}$ such that $a_n \leq \beta$ for all n sufficiently large. Prove that $\alpha \leq \beta$ WITHOUT using the Pinching Theorem.

Proof : Suppose that $\alpha > \beta$. So $\alpha - \beta > 0$. Since the sequence $\langle a_n \rangle_{n \geq 1}$ converges to α, we have

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1 : |a_n - \alpha| < \alpha - \beta$$

Since $a_n \leq \beta$ for all n sufficiently large, we know that

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2 : a_n \leq \beta$$

Pick $n \geq \max\{N_1, N_2\}$. Then $a_n \leq \beta$ and $|a_n - \alpha| < \alpha - \beta$. Hence

$$-(\alpha - \beta) < a_n - \alpha < \alpha - \beta$$

So $\beta < a_n$, a contradiction.
Hence $\alpha \leq \beta$. □

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11. Let $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ such that $\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0$. Put $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Use the Mean Value Theorem (or Rolle's Theorem) to prove that $P(a) = 0$ for some $a \in (0, 1)$.

Proof : Put

$$f(x) = \int_0^x P(t)\,dt = \int_0^x (a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0)\,dt = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0 x$$

By the Fundamental Theorem of Calculus, we have that $f'(x) = P(x)$. Clearly, f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Also, $f(0) = 0$ and $f(1) = \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0$. Hence it follows from the Mean Value Theorem that

$$0 = \frac{0 - 0}{1 - 0} = \frac{f(1) - f(0)}{1 - 0} = f'(a) = P(a)$$

for some $a \in (0, 1)$. □

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12. Let $A, B \subseteq \mathbb{R}$ be compact such that $A \cap B = \emptyset$. Prove that there exists $\delta > 0$ such that $|a - b| \geq \delta$ for all $a \in A$ and all $b \in B$.

Proof : Put $S = \{|a - b| \mid a \in A, b \in B\}$ and $\delta = \inf S$. Then we have

1. $\forall s \in S : \delta \leq s$
2. $\forall \epsilon > 0 : \exists s \in S : s < \delta + \epsilon$

Hence we get

$$\forall n \in \mathbb{N} : \exists s_n \in S : \delta \leq s_n < \delta + \frac{1}{n}$$

Since $\lim_{n \to +\infty} \delta = \lim_{n \to +\infty} \left(\delta + \frac{1}{n}\right) = \delta$, we get that $\lim_{n \to +\infty} s_n = \delta$. For all $n \geq 1$, we have that $s_n \in S$ and so $s_n = |a_n - b_n|$ for some $a_n \in A$ and some $b_n \in B$.
Since A is compact, we get that the sequence $\langle a_n \rangle_{n \geq 1}$ has a subsequence $\langle a_{n_k} \rangle_{k \geq 1}$ that converges to some element $a \in A$. Similarly, since B is compact, the sequence $\langle b_{n_k} \rangle_{k \geq 1}$ has a subsequence $\langle b_{n_{k_l}} \rangle_{l \geq 1}$ that converges to some element $b \in B$. Since $\langle a_{n_{k_l}} \rangle_{l \geq 1}$ is a subsequence of the convergent sequence $\langle a_{n_k} \rangle_{k \geq 1}$ (with limit a), we get that the sequence $\langle a_{n_{k_l}} \rangle_{l \geq 1}$ converges to a. Hence the sequence $\left\langle \left| a_{n_{k_l}} - b_{n_{k_l}} \right| \right\rangle_{l \geq 1}$ converges to $|a - b|$. But it is also a subsequence of the sequence $\langle |a_n - b_n| \rangle_{n \geq 1} = \langle s_n \rangle_{n \geq 1}$ which converges to δ. So the sequence $\left\langle \left| a_{n_{k_l}} - b_{n_{k_l}} \right| \right\rangle_{l \geq 1}$ converges to δ. Hence $\delta = |a - b|$. Clearly, $\delta \geq 0$. If $\delta = 0$ then $|a - b| = 0$ and so $a = b$, a contradiction since $a \in A$, $b \in B$ and $A \cap B = \emptyset$. So $\delta > 0$.
Pick $x \in A$ and $y \in B$. Then $|x - y| \in S$ and so $|x - y| \geq \inf S = \delta > 0$. $\square$

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be an even and monotonic function. Prove that f is constant over $\mathbb{R}$.

<u>*Proof*</u> : Pick $x > 0$. Then $-x < 0 < x$. Since f is monotonic, we get that either

$$f(-x) \le f(0) \le f(x) \quad \text{or} \quad f(-x) \ge f(0) \ge f(x)$$

But $f(-x) = f(x)$ since f is even. Hence $f(-x) = f(0) = f(x)$. So f is constant : $f(y) = f(0)$ for all $y \in \mathbb{R}$. □

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2. Prove the 'Ratio Test' :

Let $\displaystyle\sum_{n=0}^{+\infty} a_n$ be a series of real numbers such that the limit $L := \displaystyle\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists. Then

(i) the series $\displaystyle\sum_{n=0}^{+\infty} a_n$ converges absolutely if $L < 1$.

(ii) the series $\displaystyle\sum_{n=0}^{+\infty} a_n$ diverges if $L > 1$.

<u>*Proof*</u> : (a) Suppose that $L < 1$. Let $r \in \mathbb{R}$ with $L < r < 1$. Then $r - L > 0$. Since $\displaystyle\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, we get that

$$\exists N \in \mathbb{N} : \forall n \ge N : \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < r - L$$

Pick $n \ge N$. Then $\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < r - L$ and so $-(r-L) < \left|\frac{a_{n+1}}{a_n}\right| - L < r - L$. Hence $\left|\frac{a_{n+1}}{a_n}\right| < r$ and so

$$|a_{n+1}| < r|a_n| \quad \text{for all } n \ge N$$

So

$$|a_{n+2}| < r|a_{n+1}| < r \cdot r|a_n| = r^2|a_n| \quad \text{for all } n \ge N$$

Continuing this way, we get that

$$|a_{N+k}| \le r^k|a_N| \quad \text{for all } k \ge 0$$

Since the series $\displaystyle\sum_{k=0}^{+\infty} r^k|a_N|$ converges (it's a geometric series with ratio r), it follows from the Comparison Test that the series $\displaystyle\sum_{n=N}^{+\infty} |a_n| = \sum_{k=0}^{+\infty} |a_{N+k}|$ converges. Hence the series $\displaystyle\sum_{n=0}^{+\infty} a_n$ converges absolutely.

(b) Suppose $L > 1$. Let $r \in \mathbb{R}$ with $1 < r < L$. Then $L - r > 0$. Since $\displaystyle\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, we get that

$$\exists N \in \mathbb{N} : \forall n \ge N : \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < L - r$$

Pick $n \ge N$. Then $\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < L - r$ and so $-(L-r) < \left|\frac{a_{n+1}}{a_n}\right| - L < L - r$. Hence $\left|\frac{a_{n+1}}{a_n}\right| > r$ and so

$$|a_{n+1}| > r|a_n| \ge |a_n| \quad \text{for all } n \ge N$$

Hence the sequence $\langle |a_n| \rangle_{n \geq N}$ is strictly increasing. So

$$|a_n| \geq |a_{N+1}| > |a_N| \quad \text{for all } n > N \qquad (*)$$

Suppose the series $\sum_{n=0}^{+\infty} a_n$ converges. Then $\lim_{n \to +\infty} a_n = 0$ by the Zero Test. Considering the limit as $n \to +\infty$ in (*), we get that

$$0 = \lim_{n \to +\infty} a_n \geq |a_{N+1}| > |a_N| \geq 0$$

a contradiction. Hence the series $\sum_{n=0}^{+\infty} a_n$ diverges.

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3. Define the function $f : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} 1 & \text{if } x = \dfrac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

(a) Show that f is not continuous at $x = 0$.

(b) Can you alter the definition of $f(0)$ to make f continuous at $x = 0$? Justify your answer!

Proof : We will show that $\lim_{x \to 0} f(x)$ does not exist. This implies that f is not continuous at $x = 0$.

Suppose $\lim_{x \to 0} f(x)$ exists, say $\lim_{x \to 0} f(x) = L$. Since the sequence $\left\langle \frac{1}{n} \right\rangle_{n \geq 1}$ converges to 0, we get that $\left\langle f\left(\frac{1}{n}\right) \right\rangle_{n \geq 1}$ converges to L. But $f\left(\frac{1}{n}\right) = 1$ for all $n \geq 1$ and so

$$L = \lim_{n \to +\infty} f\left(\frac{1}{n}\right) = \lim_{n \to +\infty} 1 = 1$$

Since the sequence $\langle 0 \rangle_{n \geq 1}$ converges to 0, we get that $\langle f(0) \rangle_{n \geq 1}$ converges to L. But $f(0) = 0$ and so

$$L = \lim_{n \to +\infty} f(0) = \lim_{n \to +\infty} 0 = 0$$

Hence $1 = L = 0$, a contradiction.

(b) Suppose we can alter the definition of $f(0)$ to make f continuous at $x = 0$, say the function

$$g : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} f(x) & \text{if } x \neq 0 \\ L & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$ for some $L \in \mathbb{R}$. Since g is continuous at $x = 0$, we have that $\lim_{x \to 0} g(x) = g(0) = L$. Since $g(x) = f(x)$ for all $x \neq 0$, we have that $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = L$, a contradiction to (a).
Hence we can not alter the definition of $f(0)$ to make f continuous at $x = 0$.

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4. Give an ϵ-δ proof of the fact that the real function $f(x) = \dfrac{1}{x}$ is continuous at $x = 2$.

Proof : Note that the domain of f is $\mathbb{R} \setminus \{0\}$ and $f(2) = \dfrac{1}{2}$. Hence we need to prove

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} \setminus \{0\} : |x - 2| < \delta \implies \left| \frac{1}{x} - \frac{1}{2} \right| < \epsilon$$

Pick $\epsilon > 0$. Let $\delta = \min\{1, 2\epsilon\}$. Pick $x \in \mathbb{R} \setminus \{0\}$ with $|x - 2| < \delta$. Since $\delta \leq 1$, we get that $|x - 2| < 1$ and so $-1 < x - 2 < 1$. Hence $1 < x < 3$ and so $|2x| = 2x > 2$. Hence

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{|2x|} < \frac{|x-2|}{2} < \frac{\delta}{2} \leq \frac{2\epsilon}{2} = \epsilon$$

□

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5. (a) Let $f : [a, b] \to \mathbb{R}$ be bounded. Define what it means for f to be Riemman integrable over $[a, b]$ using the notions of upper and lower sums.

(b) Use your definition of part (a) to show that the function $f : \mathbb{R} \to \mathbb{R} : x \to \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is Riemann integrable over any closed and bounded interval $[a, b]$.

Proof : Let $\mathcal{P} = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$. For $i = 1, 2, \ldots, n$, we put

$$m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$$

The lower and upper sums of f with respect to $\mathcal{P}$ are

$$\mathcal{L}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \text{and} \quad \mathcal{U}(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Note that $\mathcal{L}(f, \mathcal{P}) \leq \mathcal{U}(f, \mathcal{P})$.

f is Riemann integrable over $[a, b]$ if and only if

$$\forall \epsilon > 0 : \exists \mathcal{P} : \mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P}) < \epsilon$$

(b) Suppose first that $a \geq 0$. Pick $\epsilon > 0$. Put $\mathcal{P} = \{a, b\}$. Then $m_1 = \inf\{f(x) \,|\, x \in [a, b]\} = 1 = \sup\{f(x) \,|\, x \in [a, b]\} = M_1$. Hence $\mathcal{L}(f, \mathcal{P}) = \mathcal{U}(f, \mathcal{P}) = b - a$ and so $0 = \mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P}) < \epsilon$.

Suppose next that $b < 0$. Pick $\epsilon > 0$. Put $\mathcal{P} = \{a, b\}$. Then $m_1 = \inf\{f(x) \,|\, x \in [a, b]\} = 0 = \sup\{f(x) \,|\, x \in [a, b]\} = M_1$. Hence $\mathcal{L}(f, \mathcal{P}) = \mathcal{U}(f, \mathcal{P}) = 0$ and so $0 = \mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P}) < \epsilon$.

Suppose finally that $a < 0 \leq b$. Pick $\epsilon > 0$. Choose $a < c < 0$ with $|c| < \epsilon$. Put $\mathcal{P} = \{a, c, 0, b\}$. We easily get that

$$m_1 = 0 \ , \ m_2 = 0 \ , \ m_3 = 1 \ , \ M_1 = 0 \ , \ M_2 = 1 \ \text{ and } M_3 = 1$$

Hence

$$\mathcal{L}(f, \mathcal{P}) = 0 \cdot (c - a) + 0 \cdot (0 - c) + 1 \cdot (b - 0) = b \quad \text{and} \quad \mathcal{U}(f, \mathcal{P}) = 0 \cdot (c - a) + 1 \cdot (0 - c) + 1 \cdot (b - 0) = b - c$$

Hence $\mathcal{U}(f, \mathcal{P}) - \mathcal{L}(f, \mathcal{P}) = b - c - b = -c = |c| < \epsilon$.

So f is Riemman integrable over $[a, b]$. □

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6. Let $A \subseteq \mathbb{R}$ and for all $n \geq 1$, let $f_n : A \to \mathbb{R}$ be a function that is uniformly continuous on A. Suppose that the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on A. Prove that f is uniformly continuous on A.

Proof : We need to show

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in A : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Pick $\epsilon > 0$. Since the sequence $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on A, we have

$$\exists N \in \mathbb{N} : \forall n \geq N, \forall x \in A : |f_n(x) - f(x)| < \frac{\epsilon}{3} \qquad (*)$$

Since f_N is uniformly continuous on A, we get

$$\exists \delta > 0 : \forall x, y \in A : |x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \qquad (**)$$

Pick $x, y \in A$ with $|x - y| < \delta$. The it follows from (*) and (**) that

$$\begin{aligned} |f(x) - f(y)| &= |(f(x) - f_N(x)) + (f_N(x) - f_N(y)) + (f_N(y) - f(y))| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

□

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7. Prove that $e^x > 7(x - 1)$ for all $x \geq 2$.

Proof :
Put $f(x) = e^x - 7x$. We need to prove that $f(x) > -7$ for all $x \geq 2$. Note that

$$e^2 > 2.7^2 = 7.29 > 7$$

We easily get that $f'(x) = e^x - 7$. Hence $f'(x) \geq e^2 - 7 > 0$ for all $x \geq 2$. So f is increasing on $[2, +\infty)$. Hence

$$f(x) \geq f(2) = e^2 - 14 > 7 - 14 = -7 \text{ for all } x \geq 2.$$

□

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8. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b) = 0$ and that there exists $c \in (a, b)$ such that $f(c) > 0$. Prove there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) < 0 < f'(x_2)$.

Proof : Note that f is continuous on $[a, c]$ and differentiable on (a, c). It follows from the Mean Value Theorem that

$$\frac{f(c) - f(a)}{c - a} = f'(x_2) \quad \text{for some } x_2 \in (a, c)$$

Since $f(c) > 0$, $f(a) = 0$ and $c > a$, we get that $f'(x_2) > 0$.
Similarly, we have that f is continuous on $[c, b]$ and differentiable on (c, b). It follows from the Mean Value Theorem that

$$\frac{f(b) - f(c)}{b - c} = f'(x_1) \quad \text{for some } x_1 \in (c, b)$$

Since $f(c) > 0$, $f(b) = 0$ and $b > c$, we get that $f'(x_1) < 0$. □

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9. Let $E \subseteq \mathbb{R}$ and $f, g, f_n, g_n : E \to \mathbb{R}$ be functions for all $n \geq 1$ such that $\langle f_n \rangle_{n \geq 1}$ converges uniformly to f on E and $\langle g_n \rangle_{n \geq 1}$ converges uniformly to g on E. Suppose that f and g are bounded on E. Prove that $\langle f_n g_n \rangle_{n \geq 1}$ converges uniformly to fg on E.

Proof : We need to prove

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in E : |(f_n g_n)(x) - (fg)(x)| < \epsilon$$

Pick $\epsilon > 0$. Since f and g are bounded on E, we get

$$\exists M_1 > 0 : \forall x \in E : |f(x)| \leq M_1$$

$$\exists M_2 > 0 : \forall x \in E : |g(x)| \leq M_2$$

Since the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on E and the sequence $\langle g_n \rangle_{n\geq 1}$ converges uniformly to g on E, we have

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, \forall x \in E : |f_n(x) - f(x)| < \frac{\epsilon}{2(M_2+1)}$$
$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, \forall x \in E : |g_n(x) - g(x)| < 1$$
$$\exists N_3 \in \mathbb{N} : \forall n \geq N_3, \forall x \in E : |g_n(x) - g(x)| < \frac{\epsilon}{2M_1}$$

Put $N = \max\{N_1, N_2, N_3\}$. Pick $n \geq N$ and $x \in E$. Since $n \geq N_2$, we have that $|g_n(x) - g(x)| < 1$. Hence

$$|g_n(x)| - |g(x)| \leq ||g_n(x)| - |g(x)|| \leq |g_n(x) - g(x)| < 1$$

So

$$|g_n(x)| < |g(x)| + 1 \leq M_2 + 1$$

Hence

$$\begin{aligned} |(f_n g_n)(x) - (fg)(x)| &= |[f_n(x) - f(x)]g_n(x) + f(x)[g_n(x) - g(x)]| \\ &\leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq \frac{\epsilon}{2(M_2+1)} \cdot (M_2+1) + M_1 \cdot \frac{\epsilon}{2M_1} \\ &= \epsilon \end{aligned}$$

□

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10. Evaluate $\lim_{n\to+\infty} \int_0^1 \cos\left(\frac{x^2}{n}\right) dx$.

Proof : For $n \geq 1$, put $f_n : [0,1] \to \mathbb{R} : x \to \cos\left(\frac{x^2}{n}\right)$. Put $f : [0,1] \to \mathbb{R} : x \to 1$. We prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$. We need to show :

$$\forall \epsilon > 0 : \exists N > 0 : \forall n \geq N, \forall x \in [0,1] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Since the sequence $\lim_{n\to+\infty} \frac{1}{n} = 0$ and the function $\cos t$ is continuous at $t = 0$, we have that

$$\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1$$

Hence

$$\exists N \in \mathbb{N} : \forall n \geq N : \left|\cos\left(\frac{1}{n}\right) - 1\right| < \epsilon$$

Pick $n \geq N$ and $x \in [0,1]$. Since $0 \leq \frac{x^2}{n} \leq \frac{1}{n} \leq 1$ and the function $\cos t$ is decreasing on the interval $[0,1]$, we get that

$$\cos\left(\frac{1}{n}\right) \leq \cos\left(\frac{x^2}{n}\right) \leq \cos 0 = 1$$

Hence

$$\left|\cos\left(\frac{x^2}{n}\right) - 1\right| = 1 - \cos\left(\frac{x^2}{n}\right) \leq 1 - \cos\left(\frac{1}{n}\right) < \epsilon$$

Since the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$ and f_n is continuous (and hence Riemann integrable) over $[0,1]$, we have that

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to+\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx$$

Hence

$$\lim_{n\to+\infty}\int_0^1 \cos\left(\frac{x^2}{n}\right)\,dx = \int_0^1 1\,dx = [x]_0^1 = 1 \qquad \square$$

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11. Let $f:[0,1]\to\mathbb{R}$ be continuous. Prove that $\lim\limits_{n\to+\infty}\int_0^1 x^n f(x)\,dx = 0$.

<u>*Proof*</u> : Since f is continuous on $[0,1]$ and $[0,1]$ is compact, we have that f is bounded on $[0,1]$. So

$$\exists M>0 : \forall x\in[0,1] : |f(x)|\le M$$

Pick $n\ge 1$. Then

$$|x^n f(x)| = x^n|f(x)| \le Mx^n \quad \text{ for all } x\in[0,1]$$

Note that $x^n f(x)$ is continuous on $[0,1]$ and hence Riemann integrable over $[0,1]$. So

$$0\le\left|\int_0^1 x^n f(x)\,dx\right| \le \int_0^1 |x^n f(x)|\,dx \le \int_0^1 Mx^n\,dx = \left[\frac{Mx^{n+1}}{n+1}\right]_0^1 = \frac{M}{n+1} \quad \text{ for all } n\ge 1$$

Since $\lim\limits_{n\to+\infty} 0 = \lim\limits_{n\to+\infty}\dfrac{M}{n+1} = 0$, it follows from the Pinching Theorem that $\lim\limits_{n\to+\infty}\left|\int_0^1 x^n f(x)\,dx\right| = 0$. Hence

$$\lim_{n\to+\infty}\int_0^1 x^n f(x)\,dx = 0 \qquad \square$$

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12. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. Define $g:\mathbb{R}\to\mathbb{R}: x\to \int_{x^2}^{x^3} f(t+x)\,dt$. Calculate g'.

<u>*Proof*</u> : Making the substitution $u=t+x$ (so the old variable is t and the new variable is u), we easily get (using some properties of integrals) that

$$\begin{aligned} g(x) &= \int_{x^2}^{x^3} f(t+x)\,dt \\ &= \int_{x^2+x}^{x^3+x} f(u)\,du \\ &= \int_{x^2+x}^{0} f(u)\,du + \int_0^{x^3+x} f(u)\,du \\ &= -\int_0^{x^2+x} f(u)\,du + \int_0^{x^3+x} f(u)\,du \\ &= -F(x^2+x) + F(x^3+x) \end{aligned}$$

where $F(x)=\int_0^x f(u)\,du$ for all $x\in\mathbb{R}$.
It follows from the Fundamental Theorem of Calculus that $F'(x)=f(x)$ for all $x\in\mathbb{R}$. Using the Chain Rule, we get

$$g'(x) = -F'(x^2+x)\cdot(x^2+x)' + F'(x^3+x)\cdot(x^3+x)' = -(2x+1)f(x^2+x) + (3x^2+1)f(x^3+x) \qquad \square$$

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1. Consider the sequence $\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{2a_n + 3} \quad \text{if } n \geq 1 \end{cases}$

(a) Use induction on n to prove that $0 \leq a_n \leq 3$ for all $n \geq 1$.

(b) Prove that $\langle a_n \rangle_{n \geq 1}$ is an increasing sequence.

(c) Deduce that $\langle a_n \rangle_{n \geq 1}$ converges. Find $\lim_{n \to +\infty} a_n$.

Proof : (a) Clearly, $0 \leq a_1 \leq 3$. Assume that $0 \leq a_n \leq 3$ for $n = 1, 2, \ldots, k$ for some $k \geq 1$. Then

$$a_{k+1} = \sqrt{2a_k + 3} \geq 0$$

and

$$a_{k+1} \leq 3 \iff \sqrt{2a_k + 3} \leq 3 \iff 2a_k + 3 \leq 9 \iff 2a_k \leq 6 \iff a_k \leq 3$$

(b) Pick $n \geq 1$. Then

$$a_n \leq a_{n+1} \iff a_n \leq \sqrt{2a_n + 3} \iff a_n^2 \leq 2a_n + 3 \iff a_n^2 - 2a_n - 3 \leq 0 \leq (a_n - 3)(a_n + 1) \leq 0 \iff -1 \leq a_n \leq 3$$

Hence it follows from (a) that the sequence $\langle a_n \rangle_{n \geq 1}$ is increasing.

(c) It follows from (a) and (b) that the sequence $\langle a_n \rangle_{n \geq 1}$ is an increasing sequence bounded above by 3. Hence the sequence $\langle a_n \rangle_{n \geq 1}$ converges, say to L (so $L = \lim_{n \to +\infty} a_n$). Since

$$a_{n+1} = \sqrt{2a_n + 3} \quad \text{for all } n \geq 1$$

we can apply the limit as $n \to +\infty$ to both sides. We get

$$L = \sqrt{2L + 3}$$

Solving this for L (note that $L \geq 0$) we easily get that $L = 3$. So $\lim_{n \to +\infty} a_n = 3$. □

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2. Prove that $\sqrt[3]{2}$ is irrational.

Proof : Suppose that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = \dfrac{a}{b}$ where $a, b \in \mathbb{N}_0$ and $\gcd(a, b) = 1$. Hence

$$2 = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3} \quad \text{and so} \quad 2b^3 = a^3$$

Thus $2|a^3$. So $2|a$. Put $a = 2k$ with $k \in \mathbb{N}$. Then

$$2b^3 = a^3 = (2k)^3 = 8k^3 \quad \text{and so} \quad b^3 = 4k^3$$

Hence $2|b^3$ and so $2|b$, a contradiction since $2|a$, $2|b$ but $\gcd(a, b) = 1$.
Hence $\sqrt[3]{2}$ is irrational. □

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3. Prove or give a counterexample : If $\langle F_n \rangle_{n\geq 1}$ is a sequence of closed subsets of $\mathbb{R}$, then $\bigcup_{n=1}^{+\infty} F_n$ is closed.

<u>*Proof*</u> : **FALSE** : Put $F_n = \left[\frac{1}{n}, 1\right]$ for all $n \geq 1$. Then F_n is closed for all $n \geq 1$. But

$$\bigcup_{n=1}^{+\infty} F_n = \bigcup_{n=1}^{+\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$$

which is not closed. □

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4. Define $g : \mathbb{R} \to \mathbb{R} : g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases}$

Find (with proof) all the points at which g is continuous.

<u>*Proof*</u> : First, we prove that g is continuous at $x = 0$. Since $g(0) = 0$, we need to prove

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} : |x - 0| < \delta \implies |g(x) - 0| < \epsilon$$

Pick $\epsilon > 0$. Put $\delta = \epsilon$. Pick $x \in \mathbb{R}$ with $|x| < \delta$. If x is irrational then $g(x) = 0$ and so $|g(x)| = |0| = 0 < \epsilon$; if x is rational then $g(x) = x$ and so $|g(x)| = |x| < \delta = \epsilon$.

Next, we prove that g is discontinuous at every other point. So pick $a \neq 0$. Then we can find a sequence of rational numbers $\langle a_n \rangle_{n\geq 1}$ that converges to a and a sequence of irrational numbers $\langle b_n \rangle_{n\geq 1}$ that converges to a.
Suppose that g is continuous at a. Since $\lim_{n\to+\infty} a_n = a$ and g is continuous at a, we have that $\lim_{n\to+\infty} g(a_n) = g(a)$. But a_n is rational and so $g(a_n) = a_n$ for all $n \geq 1$. So $g(a) = \lim_{n\to+\infty} g(a_n) = \lim_{n\to+\infty} a_n = a$. Since $\lim_{n\to+\infty} b_n = a$ and g is continuous at a, we have that $\lim_{n\to+\infty} g(b_n) = g(a)$. But b_n is irrational and so $g(b_n) = 0$ for all $n \geq 1$. So $g(a) = \lim_{n\to+\infty} g(b_n) = \lim_{n\to+\infty} 0 = 0$. We get that $a = g(a) = 0$, a contradiction.
Hence g is not continuous at a. □

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5. Let $X, Y \subset \mathbb{R}$ and $f : X \to Y$ a function. For $B \subseteq Y$, we define $f^{-1}[B] = \{x \in X \mid f(x) \in B\}$.
Let I be an index set and $B_i \subseteq Y$ for all $i \in I$. Prove that $f^{-1}\left[\bigcap_{i\in I} B_i\right] = \bigcap_{i\in I} f^{-1}[B_i]$.

<u>*Proof*</u> : Let $x \in X$. Then we have

$$\begin{aligned} x \in f^{-1}\left[\bigcap_{i\in I} B_i\right] &\iff f(x) \in \bigcap_{i\in I} B_i \\ &\iff \forall i \in I : f(x) \in B_i \\ &\iff \forall i \in I : x \in f^{-1}[B_i] \\ &\iff x \in \bigcap_{i\in I} f^{-1}[B_i] \end{aligned}$$

So $f^{-1}\left[\bigcap_{i\in I} B_i\right] = \bigcap_{i\in I} f^{-1}[B_i]$. □

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6. (a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove the following :

Let $f : (a, b) \to \mathbb{R}$ be differentiable over (a, b). Suppose that $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant over (a, b).

Proof : (a) Let f be continuous on $[a, b]$ and differentiable on (a, b). Then $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in (a, b)$.

(b) Suppose f is not constant over (a, b). Then $f(x) \neq f(y)$ for some $a < x < y < b$. Note that f is continuous on $[x, y]$ and differentiable on (x, y). By the Mean Value Theorem, we get that $\dfrac{f(y) - f(x)}{y - x} = f'(c)$ for some $c \in (x, y)$. Since $f'(c) = 0$, we get that $f(x) = f(y)$, a contradiction.
Hence f is constant over (a, b). □

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7. If $A, B \subseteq \mathbb{R}$, we define $A + B = \{a + b \,|\, a \in A, b \in B\}$. Prove that $A + B$ is compact if A and B are compact.
hint : use the characterization of compact sets that involves sequences!

Proof : Recall that a set D is compact if and only if every sequence $\langle d_n \rangle_{n \geq 1}$ in D has a subsequence that converges to some element in D.

Let $\langle x_n \rangle_{n \geq 1}$ be a sequence in $A + B$. Then for all $n \geq 1$, we have that $x_n = a_n + b_n$ for some $a_n \in A$ and some $b_n \in B$. Since A is compact, we have that the sequence $\langle a_n \rangle_{n \geq 1}$ has a subsequence $\langle a_{n_k} \rangle_{k \geq 1}$ that converges to some element $a \in A$. Since B is compact, we get that the sequence $\langle b_{n_k} \rangle_{k \geq 1}$ has a subsequence $\langle b_{n_{k_l}} \rangle_{l \geq 1}$ that converges to some $b \in B$. Since the sequence $\langle a_{n_{k_l}} \rangle_{l \geq 1}$ is a subsequence of the convergent sequence $\langle a_{n_k} \rangle_{k \geq 1}$ (with limit a), we get that the sequence $\langle a_{n_{k_l}} \rangle_{l \geq 1}$ converges to a. Hence the sequence $\langle a_{n_{k_l}} + b_{n_{k_l}} \rangle_{l \geq 1}$ is a subsequence of the sequence $\langle a_n + b_n \rangle_{n \geq 1} = \langle x_n \rangle_{n \geq 1}$ and it converges to $a + b$, which is an element of $A + B$. So $A + B$ is compact. □

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8. Let $f : D \to \mathbb{R}$ be uniformly continuous over D and $\langle d_n \rangle_{n \geq 1}$ a Cauchy sequence with $d_n \in D$ for all $n \geq 1$. Prove that $\langle f(d_n) \rangle_{n \geq 1}$ is a Cauchy sequence.

Proof : We need to show that

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : |f(d_m) - f(d_n)| < \epsilon$$

Pick $\epsilon > 0$. Since f is uniformly continuous over D, we get

$$\exists \delta > 0 : \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \qquad (*)$$

Since the sequence $\langle d_n \rangle_{n \geq 1}$ is a Cauchy sequence, we have

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |d_m - d_n| < \delta \qquad (**)$$

Pick $m, n \in \mathbb{N}$ with $m, n \geq N$. By (**), we get that $|d_m - d_n| < \delta$. So it follows from (*) that $|f(d_m) - f(d_n)| < \epsilon$. □

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9. For $n \geq 1$, we define $f_n : [0, 1] \to \mathbb{R} : x \to \dfrac{x^2 - x}{n^2}$

(a) Find the function $f : [0, 1] \to \mathbb{R}$ such that $\langle f_n \rangle_{n \geq 1}$ converges pointwise to f.

(b) Is this convergence uniform? Prove your answer!

Proof : Put $f : [0,1] \to \mathbb{R} : x \to 0$. We prove that the sequence $\langle f_n \rangle_{n\geq 1}$ converges uniformly to f on $[0,1]$. Then the sequence $\langle f_n \rangle_{n\geq 1}$ converges of course also pointwise to f on $[0,1]$.
So we need to show

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [0,1] : |f_n(x) - f(x)| < \epsilon$$

Pick $\epsilon > 0$. Let $N \in \mathbb{N}$ with $N > \sqrt{\frac{2}{\epsilon}}$. Note that $\frac{2}{N^2} < \epsilon$. Pick $n \geq N$ and $x \in [0,1]$. Then $|x^2 - x| \leq |x^2| + |x| \leq 1 + 1 = 2$. Hence

$$|f_n(x) - f(x)| = \left| \frac{x^2 - x}{n^2} - 0 \right| = \frac{|x^2 - x|}{n^2} \leq \frac{2}{n^2} \leq \frac{2}{N^2} < \epsilon$$ □

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10. Consider the series $\sum_{n=1}^{+\infty} \frac{x}{n^2 + x^2}$. Prove that this series converges uniformly on $[0,1]$.

Proof : Pick $n \geq 1$. Then for all $x \in [0,1]$, we have that $|x| = x \leq 1$ and $|n^2 + x^2| = n^2 + x^2 \geq n^2$; hence

$$\left| \frac{x}{n^2 + x^2} \right| \leq \frac{1}{n^2} \qquad \text{for all } x \in [0,1]$$

Since the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges (it's a p-series with $p = 2$), it follows from the Weierstrass M-test that the series $\sum_{n=1}^{+\infty} \frac{x}{n^2 + x^2}$ converges uniformly on $[0,1]$. □

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11. Let $f : [a,b] \to \mathbb{R}$ be continuous over $[0,1]$. Suppose that $f(x) \geq 0$ for all $x \in [0,1]$. Prove that

$$\left[\int_0^1 f(x)\, dx \right]^2 \leq \int_0^1 f^2(x)\, dx$$

Proof : Pick $n \geq 1$. Put $\mathcal{P}_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \right\}$. For $i = 1, \ldots, n$, pick $x_i \in \left[\frac{i-1}{n}, \frac{i}{n} \right]$. Note that f and f^2 are continuous on $[0,1]$ and hence Riemann-integrable over $[0,1]$. Since $\lim_{n\to+\infty} \|\mathcal{P}_n\| = \lim_{n\to+\infty} \frac{1}{n} = 0$, we get that

$$\int_0^1 f(x)\, dx = \lim_{n\to+\infty} \mathcal{R}(f, \mathcal{P}_n, x_n) \quad \text{and} \quad \int_0^1 f^2(x)\, dx = \lim_{n\to+\infty} \mathcal{R}(f^2, \mathcal{P}_n, x_n)$$

We easily get that

$$\mathcal{R}(f, \mathcal{P}_n, x_n) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \quad \text{and} \quad \mathcal{R}(f^2, \mathcal{P}_n, x_n) = \frac{f^2(x_1) + f^2(x_2) + \cdots + f^2(x_n)}{n}$$

Since $f(x_i) \geq 0$ for $i = 1, 2, \ldots, n$, we get that

$$\mathcal{R}^2(f, \mathcal{P}_n, x_n) = \left(\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \right)^2 \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \mathcal{R}(f^2, \mathcal{P}_n, x_n)$$

(the inequality between the arithmetic average and quadratic average). Considering the limit as $n \to +\infty$, we get

$$\left[\int_0^1 f(x)\, dx \right]^2 = \lim_{n\to+\infty} \mathcal{R}^2(f, \mathcal{P}_n, x_n) \leq \lim_{n\to+\infty} \mathcal{R}(f^2, \mathcal{P}_n, x_n) = \int_0^1 f^2(x)\, dx$$ □

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12. Let $f, g : [0,1] \to \mathbb{R}$ be continuous over $[0,1]$ such that $f(0) \leq g(0)$ and $f(1) \geq g(1)$. Prove that there exists $c \in [0,1]$ such that $f(c) = g(c)$.

Proof : Consider the function $h : [0,1] \to \mathbb{R} : x \to f(x) - g(x)$. Since f and g are continuous on $[0,1]$, we get that $h = f - g$ is continuous on $[0,1]$. Note that $h(0) = f(0) - g(0) \leq 0$ and $h(1) = f(1) - g(1) \geq 0$. It follows from the Intermediate Value Theorem that $h(c) = 0$ for some $c \in [0,1]$. Hence $f(c) = g(c)$. $\square$

Mathematics Department Qualifying Exam - Solutions **Spring 2009**
Subject : Analysis

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*Instructions : Solve **8** of the following 12 problems :*

1. (a) State the ε-δ definition of uniform continuity of a function $f : I \to \mathbb{R}$.
Solution: f is uniformly continuous on I if and only if for all $\epsilon > 0$ there exists a $\delta > 0$, such that $|x - y| < \delta, \quad x, y \in I$ implies $|f(x) - f(y)| < \epsilon$.

   (b) Consider the function $f : (0, 1) \to \mathbb{R}$ where $f(x) = x^2$ for all $x \in (0, 1)$. Use the definition you gave in (a) to prove that f is uniformly continuous over $(0, 1)$.
Solution: Let $\epsilon > 0$ be given and set $\delta = \dfrac{\epsilon}{2}$. If $x, y \in (0, 1)$ and $|x - y| < \delta$, then
$$|x^2 - y^2| = |(x + y)(x - y)| < 2|x - y| < 2\delta = \epsilon,$$
and by part (a) we conclude that $f(x) = x^2$ is uniformly continuous on $(0, 1)$.

2. Prove that the series $\displaystyle\sum_{k=1}^{+\infty} \frac{\sin(\sqrt{k}\, x)}{k^2 + x^2}$ converges uniformly over $\mathbb{R}$.
Solution: Note that
$$\left| \frac{\sin(\sqrt{k}\, x)}{k^2 + x^2} \right| \le \frac{1}{k^2 + x^2} \le \frac{1}{k^2}, \quad \forall x \in \mathbb{R}.$$
Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < +\infty$, the Weierstrass M-test applies, and we conclude that the indicated series converges uniformly on $\mathbb{R}$.

3. Let P, Q be non-empty bounded subsets of $\mathbb{R}$ such that for each $x \in P$ there exists $y \in Q$ with $x \le y$.

   (a) Show that $\sup(P) \le \sup(Q)$.
Solution: Suppose not. Then $\sup P > \sup Q$. This implies that there is an element $p \in P$ such that $\sup P \ge p > \sup Q$, which in turn gives $p > q$ for all $q \in Q$, a contradiction.

   (b) Is $\inf(P) \le \inf(Q)$? If true, prove the statement; if false, give a counterexample.
Solution: The answer is a resounding NO! For a counterexample, set $P = [0, 1]$ and $Q = [-2, 2]$.

4. Evaluate $\displaystyle\lim_{n\to+\infty} \int_{-2}^{1} e^{\frac{x^2}{n}}\, dx$. Justify your answer!
Solution: The limit is equal to 3, by interchanging the limit with the integral. One

can do this, because $e^{x^2/n} \to 1$ uniformly on $[-2,1]$. To see this, let $\epsilon > 0$ be given. Then

$$\left|e^{\frac{x^2}{n}} - 1\right| \le e^{\frac{4}{n}} - 1 < \epsilon \qquad x \in [-2,1], n \gg 1$$

since $4/n \to 0$ and the exponential function is continuous on $\mathbb{R}$.

5. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ if it exists. If it does not exist, write DNE. Prove your answer!
Solution: The limit does not exist. Along the curve $y = x^2$, we get that the quotient $\dfrac{x^2 y}{x^4 + y^2}$ is constant $1/2$. On the other hand

$$\lim_{(x,0)\to(0,0)} \frac{x^2 y}{x^4 + y^2} = 0.$$

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   (a) At which points in $\mathbb{R}$ is f continuous? Justify your answer!
   (b) At which points in $\mathbb{R}$ is f differentiable? Justify your answer!

Solution: Standard arguments (using the definitions of continuity and differentiability) show that f is continuous only at $x = 0$, and it is actually differentiable there.

7. (a) State the Mean Value Theorem.
Solution: Let f be differentiable on an open interval I. For all $a, b \in I$ with $a \neq b$, there exists a c between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable over $\mathbb{R}$ such that $f(0) = 1$ and $|f'(x)| \le 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \le |x| + 1$ for all $x \in \mathbb{R}$.
Solution: By part (a), we have

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

for some c between x and 0. It follows that for any $x \in \mathbb{R}$ we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x) - 1}{x}\right| = |f'(c)| \le 1,$$

and hence

$$|f(x) - 1| \leq |x|, \qquad \forall x \in \mathbb{R}.$$

Since $|f(x)| - 1 \leq |f(x) - 1|$, we obtain $|f(x)| \leq |x| + 1$, as desired.

8. Let $f : [0,1] \to \mathbb{R}$ be continuous and positive on $[0,1]$ such that $\int_0^1 f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [0,1]$.
Solution: Suppose not. Then there is a point $x_0 \in [0,1]$ such that $f(x_0) > 0$. By the sign preserving property of continuous functions, there exists $\epsilon > 0$, such that $f(x) > 0$ on $I \stackrel{\text{def}}{=} [x_0 - \epsilon, x_0 + \epsilon] \cap [0,1]$. Let P_I be a partition of $[0,1]$ that contains the endpoints of I. Then $L(P_I, f) > 0$, and since f is integrable on $[0,1]$, we have

$$\int_0^1 f(x)dx = L(f) = \sup_P L(P, f) \geq L(P_I, f) > 0,$$

a contradiction.

9. Put $\mathcal{C} = \left\{ \left(\frac{x}{2}, \frac{x+1}{2} \right) : 0 < x < 1 \right\}$. Show that $\mathcal{C}$ is an open cover of $(0,1)$ and that $\mathcal{C}$ does not contain a finite subcover of $(0,1)$.
Solution Given $x \in (0,1)$, it is contained in the open interval $\left(\frac{x}{2}, \frac{x+1}{2} \right)$, hence $\mathcal{C}$ is an open cover of $(0,1)$. Now if $\mathcal{C}_F$ is *any* finite subcover of $\mathcal{C}$, then there is a smallest $x_m \in (0,1)$ such that $\left(\frac{x_m}{2}, \frac{x_m + 1}{2} \right) \in \mathcal{C}_F$. Consequently, $\frac{x_m}{4}$ is not in any of the sets contained in $\mathcal{C}_F$. Thus no finite subcover of $\mathcal{C}$ can be an open cover of $(0,1)$.

10. For all $x, y > 0$ we define $d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$. Is d a metric on $(0, +\infty)$? Prove your answer!
Solution: The only mildly (and even that is a stretch) interesting property to check is the triangle inequality, as $d(x,y)$ is trivially non-negative, symmetric, and 0 if and only if $x = y$. For the triangle inequality, simply calculate

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y} \right| \leq \left| \frac{1}{x} - \frac{1}{z} \right| + \left| \frac{1}{z} - \frac{1}{y} \right| = d(x,z) + d(z,y).$$

11. Let f and g be defined on $[a,b]$ with g continuous, $f \geq 0$, and f integrable. Show that there exists a point $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = g(x_0) \int_a^b f(x)dx.$$

Solution: Since g is continuous on $[a, b]$, it attains both its minimum and its maximum there. Write $g_m \stackrel{\text{def}}{=} \min_{x\in[a,b]} g(x)$ and $g_M \stackrel{\text{def}}{=} \max_{x\in[a,b]} g(x)$. Then

$$g_m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx} \leq g_M$$

hence by the Intermediate Value Theorem, there is an point $x_0 \in [a, b]$, such that

$$g(x_0) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx},$$

and the proof is complete.

12. Consider the sequence $\langle a_n \rangle_{n\geq 1}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \dfrac{1}{a_n} \text{ for all } n \geq 1 \end{cases}$$

Prove that the sequence $\langle a_n \rangle_{n\geq 1}$ converges.
Solution: We show that a_n is bounded and monotone. We prove the first assertion by induction. We claim that $1 \leq a_n \leq \frac{3+\sqrt{5}}{2}$. Since $1 \leq a_1 = 1 \leq \frac{3+\sqrt{5}}{2}$, we have our base case. Assume now that $1 \leq a_n \leq \frac{3+\sqrt{5}}{2}$. Then

$$(\star)\ 1 < 2 \leq a_{n+1} = 3 - \frac{1}{a_n} \leq 3 - \frac{2}{3+\sqrt{5}} = \frac{7+3\sqrt{5}}{3+\sqrt{5}} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}.$$

Next we show that a_n is monotone increasing. We start by noting that $a_1 = 1 < a_2 = 2$. By the recursive formulation we see that $a_{n+1} > a_n$ if and only if

$$(\dagger) \quad a_n^2 - 3a_n + 1 < 0.$$

This happens precisely if $\frac{3-\sqrt{5}}{2} < a_n < \frac{3+\sqrt{5}}{2}$. Since $\frac{3-\sqrt{5}}{2} < 1$, the bounds exhibited in $(\star)$ assure that $(\dagger)$ holds. This completes the proof.