2009
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. The sum of the (positive integer) divisors of 2009 is ________.

   (a) 2391 (b) 2392

   (c) 2393 (d) 2394

   (e) None of these

   Solution. (d) The prime factorization of 2009 is $2009 = 7^2 \cdot 41$. Thus, the sum of the divisors of 2009 is

$$\begin{aligned}(1+7+7^2)(1+41) &= 57 \cdot 42 \\ &= 2394.\end{aligned}$$

2. The solution x to the equation $\log_2 \log_3 \log_5 x = 1$ satisfies ________.

   (a) $0 < x \leq 10$ (b) $10 < x \leq 100$

   (c) $100 < x \leq 1000$ (d) $1000 < x \leq 10000$

   (e) None of these

   Solution. (e) The solution to $\log_2 y = 1$ is $y = 2$. Hence, $2 = \log_3 \log_5 x$. The solution to $2 = \log_3 w$ is $w = 9$, hence $\log_5 x = 9$. The solution to this last equation is $x = 5^9$. Now, $5^9 = (5^4)^2 \cdot 5 = (625)^2 \cdot 5 > 100^2 = 10000$. Thus $x > 10000$, none of the answer choices provided.

3. The number of real distinct solutions to the equation

$$\frac{1}{x} - \frac{5}{x^2} + \frac{6}{x^3} = x^3 - 5x^2 + 6x$$

is ________.

(a) 1 (b) 2

(c) 3 (d) 4

(e) None of these

Solution. (d) Multiply each side of the equation by x^3, and rearrange the right hand side, to obtain the equation

$$x^2 - 5x + 6 = x^4(x^2 - 5x + 6)$$

Since $x^2 - 5x + 6 = (x-2)(x-3)$, we get the two real solutions $x = 2, 3$. Now divide the equation by $x^2 - 5x + 6$ to obtain $1 = x^4$, with the two additional solutions $x = -1, 1$. This gives us a total of 4 real solutions $(x = -1, 1, 2, 3)$.

4. The base 2 representation of the number N is $(11 \cdots 11)_2$ (that is 2009 ones). What is the base 4 representation of N?

(a) $(33 \cdots 33)_4$ (1005 threes) (b) $(33 \cdots 33)_4$ (1004 threes)

(c) $(133 \cdots 33)_4$ (1005 threes) (d) $(133 \cdots 33)_4$ (1004 threes)

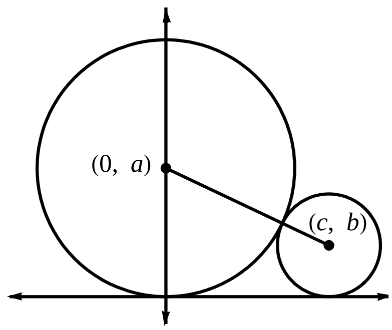
(e) None of these

Solution. (d)

$$\begin{aligned}
(11 \cdots 11)_2 &= 2^{2008} + 2^{2007} + 2^{2006} + \cdots + 2^1 + 2^0 \\
&= 2^{2008} + (2 \cdot 2^{2006} + 2^{2006}) + (2 \cdot 2^{2004} + 2^{2004}) \cdots + (2 + 1) \\
&= 4^{1004} + (2 + 1) \cdot 4^{1003} + (2 + 1) \cdot 4^{1002} + \cdots + (2 + 1) \cdot 4^0 \\
&= 4^{1004} + 3 \cdot 4^{1003} + 3 \cdot 4^{1002} + \cdots + 3 \cdot 4^0 \\
&= (133 \cdots 33)_4,
\end{aligned}$$

where there are 1004 threes.

5. In the figure below, the two circles are tangent and the respective centers $(0, a)$ and (c, b)) are connected by a line segment that goes through the point of tangency. The numbers a, b and c then satisfy the algebraic equation ________.



(a) $a^2 = 4bc$

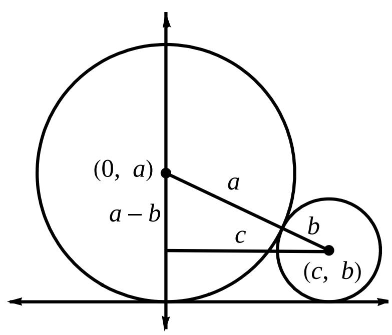
(b) $c^2 = 4ab$

(c) $a^2 + b^2 = c^2$

(d) $b^2 + c^2 = a^2$

(e) None of these

Solution. **(b)** Use the Pythagorean Theorem on the right triangle pictured.



$$(a - b)^2 + c^2 = (a + b)^2 \implies -2ab + c^2 = 2ab$$
$$\implies c^2 = 4ab.$$

6. Lenny sells the *ispud*©. Last week he sold 1000 *ispud*©s. Lenny wants to increase the price of an *ispud*© by 20%. What is the least number

he needs to sell this week at the increased price to make at least as much income as he made last week?

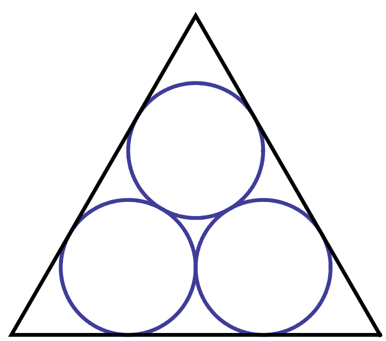
(a) 834 (b) 835

(c) 836 (d) 837

(e) None of these

Solution. **(a)** Let C be the cost of the *ispud*© last week. Then Lenny's income from last week's sales is $1000C$. If Lenny sells N *ispud*©s this week at the price $(1.2)C$, then to obtain a comparable income, it must be the case that $N(1.2)C \geq 1000C$. Thus, $N \geq 1000/1.2 = 833\frac{1}{3}$. Thus, the least number Lenny must sell is 834.

7. The equilateral triangle is circumscribing (tangent to) the three kissing (tangent) unit radius (radius = 1) circles. The area of the triangle is ________.

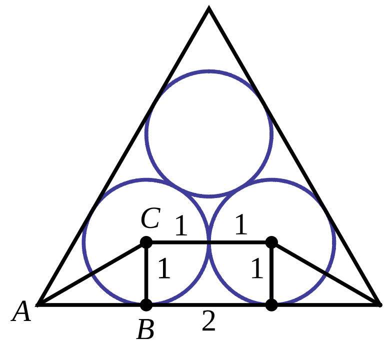


(a) $4 + 6\sqrt{3}$ (b) $6 + 4\sqrt{3}$

(c) $3 + 5\sqrt{3}$ (d) $5 + 3\sqrt{3}$

(e) None of these

Solution. **(b)** The area enclosed by an equilateral triangle of side length s is $(\sqrt{3}/4)s^2$. So we need only determine the side length of the triangle. Note that $\triangle ABC$ in the figure below is a 30°-60°-90° triangle, and since $BC = 1$, it follows that $AB = \sqrt{3}$.



Thus the side length of the circumscribing triangle is $s = 2+2\sqrt{3}$. The area is then

$$\begin{aligned} \text{Area} &= \frac{\sqrt{3}}{4}s^2 \\ &= \frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 \\ &= \sqrt{3}(1+\sqrt{3})^2 \\ &= \sqrt{3}(4+2\sqrt{3}) \\ &= 6+4\sqrt{3}. \end{aligned}$$

8. Let $f(x) = x^{2009}+x^{1776}$. Then $f(-1) = 0$, and hence $f(x) = (x+1)q(x)$ for some polynomial $q(x)$. Determine $q(-1)$.

(a) 0

(b) 233

(c) 1776

(d) 1

(e) None of these

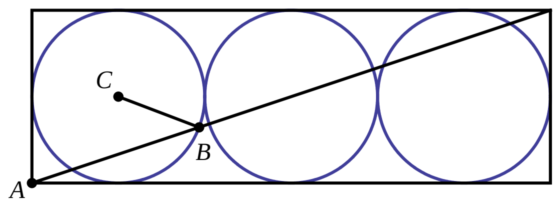
Solution. (b) We may write

$$\begin{aligned} q(x) &= \frac{f(x)}{x+1} \\ &= \frac{x^{2009}+x^{1776}}{x+1} \\ &= \frac{x^{1776}(x^{233}+1)}{x+1} \end{aligned}$$

$$= \quad x^{1776}(x^{232} - x^{231} + x^{230} - x^{229} + \cdots - x^1 + 1),$$

where, in the second factor, the coefficient of x^n is 1 if n is even and -1 if n is odd. It follows that $q(-1) = (-1)^{1776}(\underbrace{1 + 1 + \cdots + 1 + 1}_{233}) = 233$.

9. The three mutually tangent circles, all the same radius, are circumscribed by the rectangle. The diagonal of the rectangle intersects the left-most circle at the indicated point B. Let C be the center of the left-most circle and let A be the lower left-hand corner of the rectangle. Then $\sin \angle ABC =$ ________.



(a) $\dfrac{\sqrt{10}}{5}$

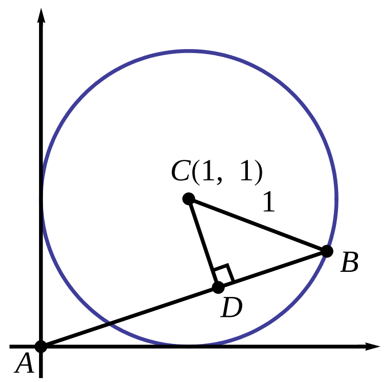
(b) $\dfrac{3}{5}$

(c) $\dfrac{2}{3}$

(d) $\dfrac{4}{5}$

(e) None of these

Solution. (a) We may assume the circles all have unit radius. Place the figure in a rectangular coordinate system where $A = (0, 0)$. Drop a perpendicular from C to a point $D \in \overline{AB}$. Then we have the following picture.



Then $BC = 1$ and so $\sin \angle ABC = CD$. The equation of the line $\overline{AB}$ is $y = x/3$ and so the equation of the line $\overline{CD}$ is $y - 1 = -3(x - 1)$, which is equivalent to $y = -3x + 4$. Thus we can determine the coordinates of D by solving the pair of equations ($\overline{AB}$ and $\overline{CD}$) to get

$$D = \left(\frac{6}{5}, \frac{2}{5}\right).$$

Now we just need the distance formula to finish the problem.

$$\begin{aligned}
\sin \angle ABC &= CD \\
&= \sqrt{\left(\frac{6}{5} - 1\right)^2 + \left(\frac{2}{5} - 1\right)^2} \\
&= \sqrt{\frac{1}{25} + \frac{9}{25}} \\
&= \frac{\sqrt{10}}{5}.
\end{aligned}$$

10. There are 10 students in a room. There are 4 Freshman, 1 Sophomore, 3 Juniors, and 2 Seniors. Suppose 5 of the students are chosen at random. What is the probability that, among the chosen 5, the majority is Freshman?

(a) $\frac{11}{42}$ (b) $\frac{13}{42}$

(c) $\frac{15}{42}$ (d) $\frac{17}{42}$

(e) None of these

Solution. (a) The question is asking for the probability that of the 5 students chosen, 3 or 4 are Freshman. The number of ways to select 5 students from 10, is

$$\begin{aligned}\binom{10}{5} &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 252.\end{aligned}$$

The number of ways that you can choose 3 Freshman and 2 non-Freshman is

$$\begin{aligned}\binom{4}{3}\binom{6}{2} &= \left(\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}\right)\left(\frac{6 \cdot 5}{2 \cdot 1}\right) \\ &= 60.\end{aligned}$$

And the number of ways that you can choose 4 Freshman and 1 non-Freshman is

$$\begin{aligned}\binom{4}{4}\binom{6}{1} &= \left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}\right)\left(\frac{6}{1}\right) \\ &= 6.\end{aligned}$$

Thus, the probability of choosing mostly Freshman is

$$\begin{aligned}\frac{60+6}{252} &= \frac{66}{252} \\ &= \frac{11}{42}\end{aligned}$$

2009
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Part II Solutions

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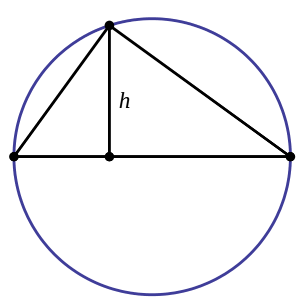
1. What is the largest area enclosed by a triangle that is inscribed in a unit circle (radius = 1) in such a way that one of the triangle sides is a diameter of the circle?

   (a) $\frac{2}{3}$ (b) $\frac{3}{4}$

   (c) $\frac{4}{5}$ (d) 1

   (e) None of these

   Solution. (d) The area of the triangle will be $\frac{1}{2} \cdot 2 \cdot h = h$, where h is the height of the triangle (see figure below).



   So the largest area will occur when $h = 1$, the radius of the circle. Hence the largest area will be equal to 1.

2. Given the base-b arithmetic identity $4214_b - 3141_b = 1043_b$, the value of the base b is ________.

   (a) 8 (b) 7

   (c) 6 (d) 5

   (e) None of these

   Solution. (b) By definition, $4214_b = 4b^3 + 2b^2 + b + 4$, $3141_b = 3b^3 + b^2 + 4b + 1$ and $1043_b = b^3 + 4b + 3$. So the identity $4214_b - 3141_b = 1043_b$ translates to $(4b^3 + 2b^2 + b + 4) - (3b^3 + b^2 + 4b + 1) = b^3 + 4b + 3$. Simplifying this equality gives $b^2 - 7b = 0$. The non-zero solution to this equation is $b = 7$.

3. A runner (R) and a walker (W) are positioned on diametrically opposite positions on a circular track. On the track, R is twice as fast as W. Assume they start simultaneously, move at constant velocity on the circular track, never change directions while R runs counterclockwise and W walks clockwise. How many times do R and W meet on the track in the time it takes W to go around the track two times?

   (a) 4 (b) 6

   (c) 8 (d) 10

   (e) None of these

   Solution. (b) Superimpose a 12-hour clock on the track. R starts at position 12 and W starts at position 6. Since R moves twice as fast as W, they meet in following positions (in order): 8, 12, 4, 8, 12, 4–which is 6 times.

4. Let T be the area of an isosceles right triangle and let C be the area of the corresponding inscribed circle to this triangle. Then, $C/T =$ ________.

(a) $(3-2\sqrt{2})\pi$

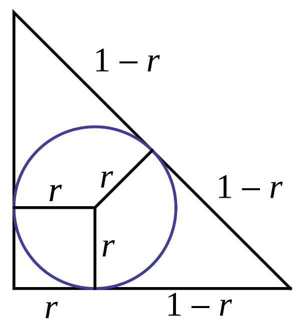
(b) $\dfrac{(4-2\sqrt{2})\pi}{5}$

(c) $\dfrac{(5-2\sqrt{2})\pi}{8}$

(d) $\dfrac{(5-2\sqrt{2})\pi}{10}$

(e) None of these

Solution. **(a)** Since we are computing the ratio of areas, we may assume the two leg lengths of the isosceles right triangle are both equal to 1. Let r be the radius of the inscribed circle. The situation, with the indicated lengths, is pictured below.



Since the hypotenuse of the right triangle is $\sqrt{2}$, we must have $2-2r=\sqrt{2}$. Solving for r gives

$$r = 1 - \frac{1}{\sqrt{2}}.$$

Thus,

$$\begin{aligned}\frac{C}{T} &= \frac{\pi r^2}{\frac{1}{2}\cdot 1\cdot 1} \\ &= 2\left(1-\frac{1}{\sqrt{2}}\right)^2\pi \\ &= (3-2\sqrt{2})\pi.\end{aligned}$$

5. Assume the two parabolas $y = a_1x^2 + b_1x + c_1$, and $y = a_2x^2 + b_2x + c_2$ (with $a_1 \neq$ and $a_2 \neq 0$) share the same vertex. Then, it is *necessary* that...

(a) $a_1 + b_1 = a_2 + b_2$

(b) $a_1 + b_2 = a_2 + b_1$

(c) $a_1 b_2 = a_2 b_1$

(d) $a_1 b_1 = a_2 b_2$

(e) None of these

Solution. **(c)** Completing the square, we get

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \end{aligned}$$

Thus, the vertex of the parabola $y = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right).$$

So, if $y = a_1x^2 + b_1x + c_1$, and $y = a_2x^2 + b_2x + c_2$ share the same vertex, then

$$-\frac{b_1}{2a_1} = -\frac{b_2}{2a_2} \qquad \text{and} \quad -\frac{b_1^2}{4a_1} + c_1 = -\frac{b_2^2}{4a_2} + c_2.$$

The first equality gives $a_1b_2 = a_2b_1$, which gives a necessary condition. The other three answer choices can be eliminated using the two parabolas $y = 4x^2 - 8x$ and $y = x^2 - 2x - 3$, which share the same vertex $(1, -4)$.

6. Simplify the product

$$(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8).$$

(a) $\log_{27} 33$

(b) 1

(c) $\log_2 163$

(d) 4

(e) None of these

Solution. **(e)** Convert all of the logarithms to base 2 using the change-of-base formula:

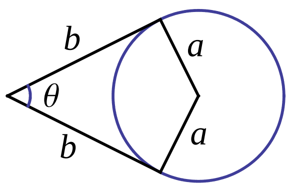
$$\log_b a = \frac{\log_2 a}{\log_2 b}.$$

Then,

$$\begin{aligned}(\log_2 3)(\log_3 4)(\log_4 5) \log_5 6)(\log_6 7)(\log_7 8) &= \log_2 3 \cdot \frac{\log_2 4}{\log_2 3} \cdot \frac{\log_2 5}{\log_2 4} \cdot \frac{\log_2 6}{\log_2 5} \cdot \frac{\log_2 7}{\log_2 6} \cdot \frac{\log_2 8}{\log_2 7} \\ &= \log_2 8 \\ &= 3,\end{aligned}$$

none of the answer choices provided.

7. In the picture below, the two line segments of length b are each tangent to the circle of radius a and meet in an angle θ. Then, $\sin\theta =$ ______.



(a) $\dfrac{ab}{a^2+b^2}$

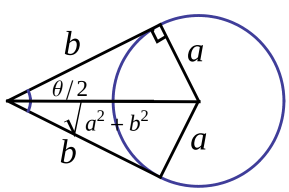
(b) $\dfrac{ab}{\sqrt{a^2+b^2}}$

(c) $\dfrac{1}{\sqrt{a^2+b^2}}$

(d) $\dfrac{1}{a^2+b^2}$

(e) None of these

Solution. **(e)** The segment that joins the intersection of the two tangent segments and the center of the circle bisects the angle θ. Also, the tangent is perpendicular to the radius, forming the right triangle pictured below.



Finally, use the double angle formula for sine,

$$\begin{aligned}\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= 2\cdot\frac{a}{\sqrt{a^2+b^2}}\cdot\frac{b}{\sqrt{a^2+b^2}} \\ &= \frac{2ab}{a^2+b^2},\end{aligned}$$

none of the answer choices provided.

8. Upon dividing 99^{2009} by 100, the remainder is ______.

(a) 1

(b) 99

(c) 9

(d) 19

(e) None of these

Solution. **(b)** Use the binomial expansion,

$$\begin{aligned}99^{2009} &= (100-1)^{2009} \\ &= \sum_{n=0}^{2009}\binom{2009}{n}100^{2009-n}(-1)^n.\end{aligned}$$

Note that every term in the sum is a multiple of 100 except the last term, which is -1. So, upon dividing by 100, the remainder will be $100-1=99$.

9. If it takes x painters to paint a wall in y days (all at the same rate), how many days does it take z painters to paint the same wall?

(a) $\dfrac{xy}{z^2}$

(b) $\dfrac{yz}{x^2}$

(c) $\dfrac{xz}{y}$

(d) $\dfrac{xy}{z}$

(e) None of these

Solution. **(d)** One painter can paint $\frac{1}{x}$ of the wall in y days. And so, one painter can paint $\frac{1}{xy}$ of the wall in one day. Consequently, z

painters can paint $\frac{z}{xy}$ of the wall in one day. So it would take z painters $\frac{xy}{z}$ days to paint the wall.

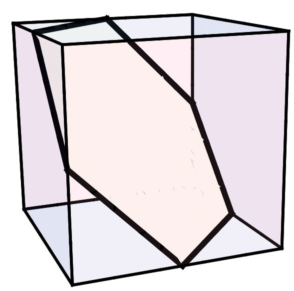
10. A plane intersects a $1' \times 1' \times 1'$ cube in a regular hexagon. What is the area of the hexagon? (A regular hexagon is a six sided polygon, all of whose side lengths are equal in measure.)

(a) $\frac{3\sqrt{3}}{4}$ ft^2 (b) $\frac{3\sqrt{2}}{4}$ ft^2

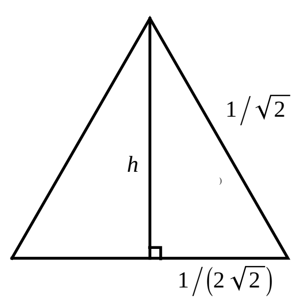
(c) $\sqrt{3}$ ft^2 (d) $\sqrt{2}$ ft^2

(e) None of these

Solution. **(a)** In order for the intersection to be a regular hexagon, the plane must intersect six of the edges at midpoints.



By the Pythagorean Theorem, each side length of the hexagon is $\frac{1}{\sqrt{2}}$ feet. To compute the area of the hexagon, divide it into six congruent equilateral triangles.



Again, by the Pythagorean Theorem, the altitude of this triangle is

$$h = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2\sqrt{2}}\right)^2} = \frac{\sqrt{3}}{2\sqrt{2}}.$$

Hence, the area of the triangle is (in square feet)

$$\text{Triangle Area} = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{8}.$$

So the hexagon area is (in square feet)

$$\text{Hexagon Area} = 6 \cdot \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}.$$