# First Annual High School Problem Solving Contest
## Department of Mathematics

**Problem 1** (10 points)

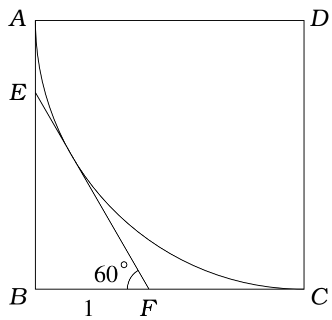
A rising number, such as 34689, is a positive integer each digit of which is larger than each of the digits to its left. When all five-digit rising numbers are arranged from smallest to largest, find the 100th number in the list.

**Solution:**

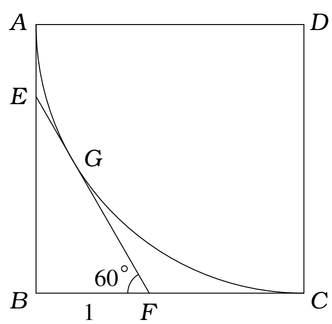
First observe that the number of five-digit rising numbers is equal to the number of ways to choose 5 different digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, since there is exactly one rising number for each choice of 5 different digits. So there are $\binom{9}{5} = \frac{9!}{5!4!} = 126$ five-digit rising numbers. Of these, $\binom{8}{4} = \frac{8!}{4!4!} = 70$ start with 1, $\binom{7}{4} = \frac{7!}{4!3!} = 35$ start with 2, $\binom{6}{4} = \frac{6!}{4!2!} = 15$ start with 3, $\binom{5}{4} = 5$ start with 4, and $\binom{4}{4} = 1$ number starts with 5. We need the 100th number in the list, so we need to find the sixth number from the end among those that start with 2. The last (largest) six numbers starting with 2, in decreasing order, are: 26789, 25789, 25689, 25679, 25678, 24789. Thus the 100th number in the list is 24789.

**Problem 2** (10 points)

An arc is drawn in a square with center $D$ at one of the vertices of the square and the arc is tangent to the opposite two sides of the square. The arc is also tangent to the hypothenuse of the $30° - 60° - 90°$ triangle $BFE$ as shown, where $BF = 1$. What is the radius of the circle?




**Solution:**




Let $G$ be the point of tangency of $EF$ with the circle. Then $AE = EG$ and $GF = FC$. Let the radius of the circle be $r$. Since $BF = 1$, $FC = r - 1$, so $GF = r - 1$ also. Since $\angle F = 60°$, $BE = \sqrt{3}$. Then $AE = r - \sqrt{3}$, so $EG = r - \sqrt{3}$. Thus $EF = EG + GF = 2r - \sqrt{3} - 1$. On the other hand, $EF = 2$. So $2r - \sqrt{3} - 1 = 2$. It follows that $r = \frac{3+\sqrt{3}}{2}$.

**Problem 3** (10 points)

A number $A$ has 666 threes as its digits and a number $B$ has 666 sixes as its digits. What are the digits in the product $A \times B$?

**Solution:**

$$\begin{aligned}
A &= \underbrace{333\ldots3}_{\text{666 threes}} \\
B &= \underbrace{666\ldots6}_{\text{666 sixes}} = 3 \times \underbrace{222\ldots2}_{\text{666 twos}} \\
A \times B &= \underbrace{999\ldots9}_{\text{666 nines}} \times \underbrace{222\ldots2}_{\text{666 twos}} \\
&= (10^{666} - 1) \times \underbrace{222\ldots2}_{\text{666 twos}} \\
&= \underbrace{222\ldots2}_{\text{666 twos}}\underbrace{000\ldots0}_{\text{666 zeros}} - \underbrace{222\ldots2}_{\text{666 twos}} \\
&= \underbrace{222\ldots2}_{\text{665 twos}}1\underbrace{777\ldots7}_{\text{665 sevens}}8
\end{aligned}$$

**Problem 4** (10 points)
Suppose the numbers $a_1, a_2, \ldots, a_{100}$ satisfy:

$$\begin{aligned}
&a_1 - 4a_2 + 3a_3 \geq 0\\
&a_2 - 4a_3 + 3a_4 \geq 0\\
&\vdots\\
&a_{98} - 4a_{99} + 3a_{100} \geq 0\\
&a_{99} - 4a_{100} + 3a_1 \geq 0\\
&a_{100} - 4a_1 + 3a_2 \geq 0
\end{aligned}$$

Let $a_1 = 1$. Find the values of $a_2, a_3, \ldots, a_{100}$.

**Solution:**

Note that the sum of the left-hand sides of these inequalities is 0. Since each of them is non-negative, this implies that each one of the inequalities must, in fact, be equalities. That is,

$$\begin{aligned}
&a_1 - 4a_2 + 3a_3 = 0\\
&a_2 - 4a_3 + 3a_4 = 0\\
&\vdots\\
&a_{98} - 4a_{99} + 3a_{100} = 0\\
&a_{99} - 4a_{100} + 3a_1 = 0\\
&a_{100} - 4a_1 + 3a_2 = 0
\end{aligned}$$

Rewrite them as:

$$\begin{aligned}
&a_1 - a_2 = 3(a_2 - a_3)\\
&a_2 - a_3 = 3(a_3 - a_4)\\
&\vdots\\
&a_{98} - a_{99} = 3(a_{99} - a_{100})\\
&a_{99} - a_{100} = 3(a_{100} - a_1)\\
&a_{100} - a_1 = 3(a_1 - a_2)
\end{aligned}$$

Substituting in the first equation for $a_2 - a_3$ from the second equation and then $a_3 - a_4$ from the third equation etc. gives $a_1 - a_2 = 3^{100}(a_1 - a_2)$ which implies $a_1 = a_2$. Using this in the first equation gives $a_2 = a_3$ and using that in turn in the second equation gives $a_3 = a_4$ etc. Thus, $a_1 = a_2 = a_3 = \ldots = a_{99} = a_{100} = 1$ (since $a_1 = 1$).

**Alternate Solution:** To simplify our explanation below, we will use the following notations:

$$i \oplus 1 = \begin{cases} i+1 & \text{if } 1 \le i \le 99 \\ 1 & \text{if } i = 100 \end{cases} \quad \text{and} \quad i \ominus 1 = \begin{cases} i-1 & \text{if } 2 \le i \le 100 \\ 100 & \text{if } i = 1 \end{cases}$$

Now, let $a_i$ be the largest number among $a_1, a_2, \ldots, a_{100}$. Since $a_{i\ominus 1} - 4a_i + 3a_{i\oplus 1} \ge 0$, or, equivalently, $a_{i\ominus 1} + 3a_{i\oplus 1} \ge 4a_i$, we have $a_{i\ominus 1} = a_i = a_{i\oplus 1}$. Then the inequality $a_i - 4a_{i\oplus 1} + 3a_{i\oplus 1\oplus 1} \ge 0$ implies that $a_{i\oplus 1\oplus 1} = a_i$ also, and so on, so all the numbers $a_1, a_2, \ldots, a_{100}$ are equal. Since $a_1 = 1$, $a_1 = a_2 = a_3 = \ldots = a_{99} = a_{100} = 1$.

**Problem 5** (10 points)

Prove that, if a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with integer coefficients has odd values at $x = 0$ and $x = 1$, then the equation

$$P(x) = 0$$

has no integer roots.

**Solution:**

Since, for any integers $l$ and $m$,

$$P(m) - P(l) = a_n[m^n - l^n] + a_{n-1}[m^{n-1} - l^{n-1}] + \cdots + a_1[m - l]$$

and, for $k = 1, \ldots, n$,

$$m^k - l^k = (m - l)(m^{k-1} + m^{k-2}l + \cdots + ml^{k-2} + l^{k-1}),$$

we conclude that, for any distinct integers $l$ and $m$, the integer $P(m) - P(l)$ is divisible by the integer $m - l$.

Hence, for any integers $l$ and $m$ of the same parity, the integers $P(l)$ and $P(m)$ have the same parity, too, which implies that, for any even $i$, $P(i)$ is odd since $P(0)$ is odd and, for any odd $i$, $P(i)$ is odd since $P(1)$ is odd.

Therefore, for any integer $i$, even or odd, $P(i)$ is odd and cannot be equal to 0.

**Problem 6** (10 points)

Is there a triangle, whose heights have lengths 1, $\sqrt{5}$, $1+\sqrt{5}$?

**Solution:**

Assume that such a triangle exists and let $A$ be its area.

Then the corresponding sides are

$$a_1 = \frac{2A}{1},\ a_2 = \frac{2A}{\sqrt{5}},\ a_3 = \frac{2A}{1+\sqrt{5}},$$

and, by the triangle inequality,

$$\frac{2A}{1} = a_1 < a_2 + a_3 = \frac{2A}{\sqrt{5}} + \frac{2A}{1+\sqrt{5}}.$$

Dividing through by $2A$, we arrive at

$$\frac{1}{1} < \frac{1}{\sqrt{5}} + \frac{1}{1+\sqrt{5}},$$

which is equivalent to

$$1 < \frac{\sqrt{5}}{5} + \frac{\sqrt{5}-1}{4},$$

and hence, to

$$\sqrt{625} = 25 < 9\sqrt{5} = \sqrt{81\cdot 5} = \sqrt{405},$$

which is a contradiction showing that the assumption is false, i.e., such a triangle does not exist.