Second Annual College Problem Solving Contest

Department of Mathematics

Name:

Problem	1	2	3	4	5	6	**Total**
Points	10	10	10	10	10	10	60
Score							

Note:

- There are 6 problems.
- Clearly show all the steps in your solution to earn credit.
- You have 2 hours to solve the problems.

Good Luck!!

Problem 1 (10 points)

Suppose that n is a positive integer and there is a function $f : \{1, 2, \ldots, n\} \mapsto \mathbb{R}$ satisfying

$$f(x + y) = f(x)f(y) + 1$$

for all $1 \leq x, y \leq n$ with $1 \leq x + y \leq n$.

What is the largest possible value of n?

Problem 2 (10 points)

Prove that if the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

with integer coefficients assumes the value 7 for four different integer values of x, then it cannot take the value 14 for any integer value of x.

Problem 3 (10 points)

Prove that for every odd natural n, $1^n + 2^n + \ldots + n^n$ is divisible by $1 + 2 + \ldots + n$.

Problem 4 (10 points)

Suppose that, for a function $f : [0, \infty) \to \mathbb{R}$,

$$\lim_{x\to\infty} \left[f(x) + \frac{1}{|f(x)|} \right] = 0.$$

Show that the limit

$$\lim_{x\to\infty} f(x)$$

exists and evaluate it.

Problem 5 (10 points)

2016 digits are written in a circular order. Prove that if the 2016-digit number obtained when we read these digits in clockwise direction beginning with one of the digits is divisible by 27, then if we read these digits in the same direction beginning with any other digit the new 2016-digit number that is formed is also divisible by 27.

Problem 6 (10 points)

Prove that $\sin 1 < \log_3 \sqrt{7}$.