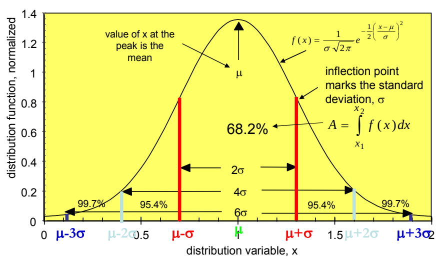
The Normal Distribution

Importance of normal or Gaussian distribution (ND)

- It is the most used distribution
- Most method are based on the assumption of ND
- Sum of many independent, random contributions variables (grain size, height, environmental factors) are of ND
- Histogram - symmetrical (mean and STD).

Normal Distribution Function



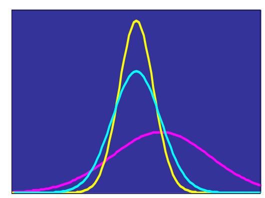
Area from $-\sigma \leq x \leq +\sigma$ ➔ 68.2 % of the total area ($x_1=-\sigma$; $x_2=\sigma$)

Area from $-2\sigma \leq x \leq +2\sigma$ ➔ 95.4% of the total area ($x_1=-2\sigma$; $x_2=2\sigma$)

Area from $-3\sigma \leq x \leq +3\sigma$ ➔ 99.7 % of the total area ($x_1=-3\sigma$; $x_2=3\sigma$)

The Gaussian Curve

- The analytical computation of the area under the Gaussian curve is difficult.
- Standardized tables generated for use.
- By varying the parameters σ and μ, we obtain different normal distributions



There are an infinite number of normal distributions!

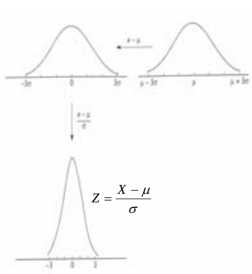
Which table to use?

Standardized Z-score

The standardization assumes a mean of zero and variance of 1.

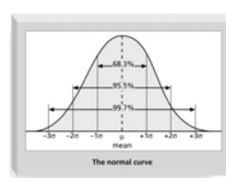
- Any random variable X with mean μ and std σ can be transformed to a standard ND using a Z score (transformation)

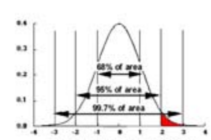
$Z = (X-\mu)/\sigma$



Advantages of using Z-score

- Permit numerical comparisons of measures from different scales by converting BOTH measures to a common scale.
- Permit numerical estimates of the expected frequency of an event
- Useful in determining the likelihood of an event occurring by chance.





Some commonly used standardized test scales...

Mean	Standard Deviation	Scale Name
500	100	SAT; GRE; LSAT; GMAT
100	15	Wechsler IQ
100	16	Stanford Binet IQ
20	5	ACT (Amer College Testing Co.)
50	10	T-scale (MMPI)

To convert raw scores to another scale, convert raw score to z score and then convert z score into the other raw score scale using the reverse transform.

Tables of Normal Distribution

- Predictions are made about ND populations by using Standard Table - Appendix in many books
  (-3 < Z < 3, 0.13% < P < 99,87%).
- The easiest answer to the standard normal distribution table is to use Excel function =NORMSDIST(Z) (for $\mu = 0$ and $\sigma=1.0$)
- The inverse function Z=NORMSINV(P)

Standard ND Tables

Area under the curve on each side of zero is 0.5.
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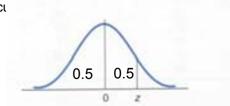


Normalization to use standard tables: $z = \frac{x-\mu}{\sigma}$

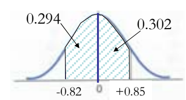
Example: if z=0.82 ➔
Area under the curve for [0, 0.82] : 0.294
Total area for [-∞, 0.82]=0.5+0.294=0.794
This value is the ***probability*** that z<0.82

Example

Company Lentil ® produces computer chip Pantsium© XIX running at 66.666 THz. However, the rival company DAM© manufactures Craplon© 66+⁺, also running at 66.666 THz. However, DAM claims that Lentil's chip is flawed, and cannot run any faster than **63** THz. **Lentil decides to test its chips**. They take a sample of 1% (1000 chips). They find that the mean speed of these chips is m =65.980 THz with a std. dev. of s =1.2 THz. Assuming that the chip speed is normally distributed, **is Lentil's speed claim justifiable**? Assume that the claim is justifiable, if 95% of the chips lie in the speed limits of 65 <x< 67 THz.

$$z = \frac{x-\mu}{\sigma} = \frac{65-65.980}{1.2} = -0.82$$

$$z = \frac{x-\mu}{\sigma} = \frac{67-65.980}{1.2} = 0.85$$



The probability that a Lentil chip has a speed in the [65, 67] THz is 0.294+0.302=0.596. Thus only 59.6% of the chips satisfy the criterion.

Now assume that the claim is justifiable, if 90% of the chips run faster than x> 65.0 THz.

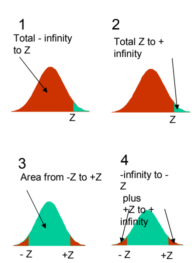
$$z = \frac{x-\mu}{\sigma} = \frac{65.0-65.980}{1.2} = -0.82$$

The probability that a Lentil chip has a speed larger than 65THz is 0.294+0.5=0.794. That means, roughly 80% of the chips satisfy the criterion.
In any case, however, Lentil does better than DAM's claim of 63 THz. What % of Lentil chips run over 63THz? (Ans. 99.3%)

$$z = \frac{x-\mu}{\sigma} = \frac{63.0-65.980}{1.2} = -2.48, \quad P(z > -2.48) = 0.493 + 0.5 = 0.993$$

Tables of Normal Distribution - 1

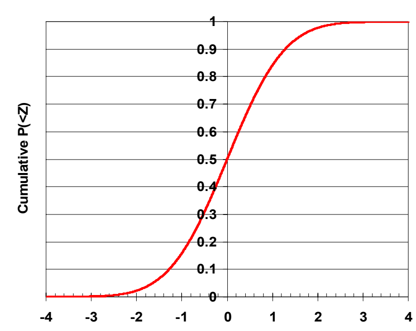
1. **P(x<Z)**
2. **P(x≥Z) = 1- P(x<Z)**
3. **P(-Z<x<Z)**
   = P(x≤Z)- P(x≤-Z)
4. **P(x≤-Z, x≥Z)**
   = 1- P(-Z<x<Z)



$$Z = \frac{X-\mu}{\sigma}$$

Standard Normal Distribution

– Cumulative curve



$$Z = \frac{X-\mu}{\sigma}$$

Distribution of the Sample Mean

Distribution of the Sample Mean

- In real life, the MEAN and STD of a population are not known
- Need to use sample means to estimate or compare with those of populations.
- To test hypotheses about a product, a value or a statement.

Sampling and Estimation

- Random Variables Used to Estimate a Population Parameter
- Sample Mean Is an Estimator of Population Mean μ

Sampling Error

- The difference between a value (a statistic) computed from a sample and the corresponding value (a parameter) computed from a population.

$$\text{Sampling Error} = \bar{x} - \mu$$

- Where:

$\bar{x}$ = **Sample mean**

μ = **Population mean**

Sampling Distribution

A *sampling distribution* is a distribution of the possible values of a statistic for a given size sample selected from a population.

Z-VALUE FOR SAMPLING DISTRIBUTION OF $\bar{x}$

$$z = \frac{(\bar{x} - \mu)}{\frac{\sigma}{\sqrt{n}}}$$

where:

$\bar{x}$ = Sample mean

μ = Population mean

σ = Population standard deviation

n = Sample size

Standard Error of Mean

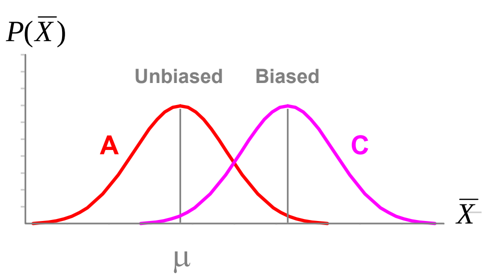
- Less Than Pop. Standard Deviation
- Formula (Sampling With Replacement)

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

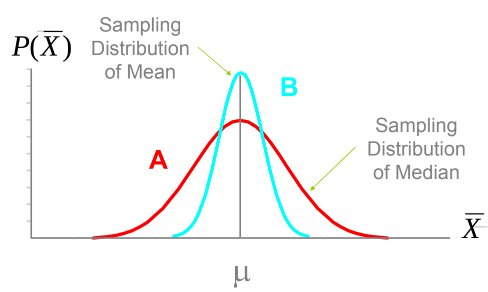
Properties of Mean

1. Unbiasedness
   - Mean of Sampling Distribution Equals Population Mean
2. Efficiency
   - Sample Mean Comes Closer to Population Mean Than Any Other Unbiased Estimator
3. Consistency
   - As Sample Size Increases, Variation of Sample Mean from Population Mean Decreases

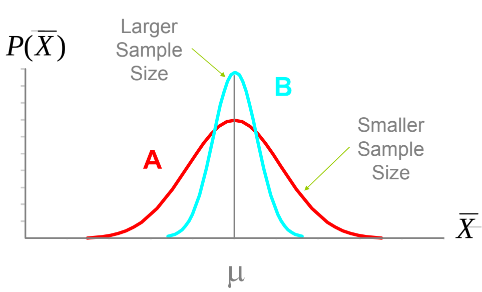
Unbiasedness



Efficiency



Consistency



Sampling from Non-Normal Populations

Central Tendency

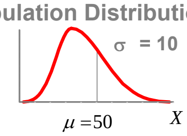
$$\mu_{\overline{X}} = \mu$$

Dispersion

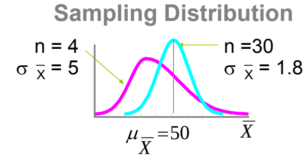
$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

Sampling **With** Replacement

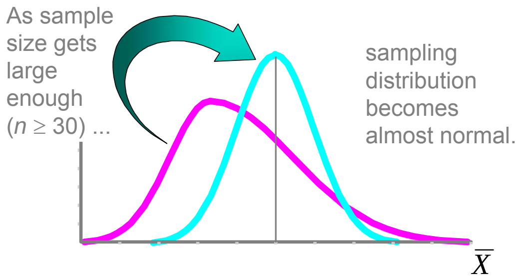
Population Distribution



Sampling Distribution



Central Limit Theorem



Central Limit Theorem

- For samples of n observations taken from a population with mean μ and standard deviation σ, **regardless of the population's distribution**, provided the sample size is sufficiently large, the distribution of the sample mean, will be normal with a mean equal to the population mean. Further, the standard deviation will equal the population standard deviation divided by the square-root of the sample size.
- The larger the sample size, the better the approximation to the normal distribution.

The **numerator of the variance equation** is called the "SUM OF SQUARES"

Short for: "the sum of the squared deviations about the arithmetic mean"

$$SS = \sum_{i=1}^{n} (X_i - \overline{X})^2$$

The sample **Standard Deviation**

- Taking the square root of the variance ***scales*** the variance back to the number scale of the original measurements...

$$STDEV = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}}$$

The Bienayme - Chebyshev Inequality

Suppose Z = k. In any population or sample of measures no more than $1/k^2$ of the raw scores can differ from the mean by k or more STDEVs... **at least $(1 - 1/k^2)$ of the values will fall within k = 1 STDEV of the mean.**

Example: ($|Z| = 4$) no more than 1/16th (6.25%) of the total scores in a sample can differ from the mean by 4 or more STDEVs .. At least $(1\text{-}1/4^2)$ or **93.75% of the values will fall within 2 STDEV of the mean.**

Generally: the larger the Z score, the rarer the event...

Sampling Distribution of mean

- Conclusions

- **Sample means are less scattered** about the overall mean than are the individual observations.
- We can **rely more on the sample means** of independent, randomly chosen specimens than on a single specimen.

Sampling from Normal Populations

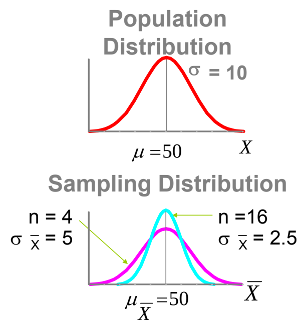
Central Tendency

$$\mu_{\bar{X}} = \mu$$

Dispersion

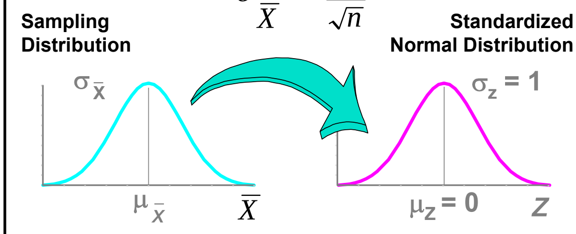
$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

Sampling **With** Replacement

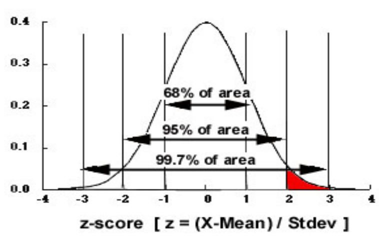


Standardizing Sampling Distribution of Mean – Z-score

$$Z = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$



Standard Normal Distribution (μ=0, σ=1)



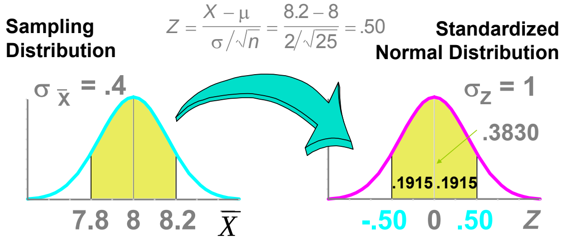
Exercise

You're an operations analyst for AT&T. Long-distance telephone calls are normally distributed with μ **= 8** min. & σ **= 2** min. If you select random samples of **25** calls, what percentage (probability) of the **sample means** would be between **7.8** & **8.2** minutes?

Example (continued)

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{7.8 - 8}{2/\sqrt{25}} = -.50$$

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{8.2 - 8}{2/\sqrt{25}} = .50$$



Normal Distribution

- Computer exercise

- Select a parameter in the data set.
- Calculate for mean and Stdev
- Find probabilities of a value that is
  - 1.25*mean or less
  - 1.2*mean or more
  - 0.75*mean or less, and
  - between (0.75-1.25)*mean