Spatial Prediction using Co-Kriging

Coregionalization and Cokriging

These two concepts are related in the same way as the *theory of regionalized variables* and (univariate) *kriging*:

1. **Coregionalization** is a theoretical model of how several variables **spatially co-vary**; this is used for . . .
2. **Cokriging** (CK), which is a method of using
   - supplementary information on a **co-variable** . . .
   - . . . to improve the **prediction** of a **target variable**.

That is, we extend the theory of ordinary kriging of regionalized variables to several variables which we show to have a **multivariate spatial cross-correlation** as well as the **univariate spatial auto-correlation**;

these are then called **co-regionalized** variables and can be used to predict.

Cokriging : Motivation

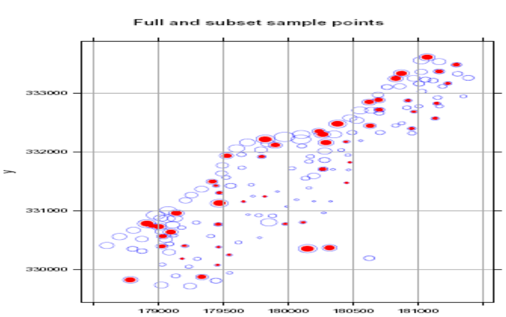
- We want to predict the values of a **target variable**
  - * often hard or expensive to measure . . .
  - * . . . so we have relatively few observations of it.
- We have, however, more values of a **second variable**, which we believe is **co-related** with the target variable; this is the **co-variable**.
  - * this is often easy and cheap to measure . . .
  - * . . . so we have many observations of it.

Strategies for using a co-variable

Co-kriging may be applicable when:

- Values of the co-variable are only used at its sample points, not needed over the whole field.
- **Co-located observations** (with the target variable) are used for *modelling* the spatial structure
- Typically there are **more observations of the co-variable**, i.e. at some points where the target variable was not measured
- **Requires a model of coregionalization**.

Two sampling intensities, some co-located points



Variograms

Two types of variograms:

1. **direct**: single regionalized variables, one variogram *per variable*
2. **cross**: *per pair* of regionalised variables

Estimating the cross-variogram

- For the two variables u and v:

$$\hat{\gamma}_{uv} = \frac{1}{2m(\vec{h})} \sum_{i=1}^{m(\vec{h})} \left\{ z_u(\vec{x}_i) - z_u(\vec{x}_i + \vec{h}) \right\} \left\{ z_v(\vec{x}_i) - z_v(\vec{x}_i + \vec{h}) \right\}$$

where $m(h)$ is the number of point-pairs separated by vector h; for an omnidirectional variogram this is the distance class

In words: if high differences between point-pairs of one variable are positively associated with high differences between point-pairs of the other variable, they will have a *high positive cross-correlation*.

This can also be a negative relation.

If the differences are randomly associated, there will be on average no spatial cross-correlation in this distance class.

Modeling the variograms

The **direct and cross-variograms must be modeled together**, with some restrictions to ensure that the resulting CK system can be solved.

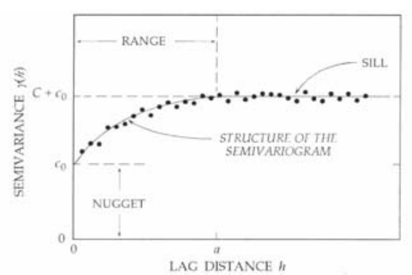
The simplest way to ensure this is to assume a **linear model of co-regionalisation**: *all variograms are linearly related*

- Same **model**, same **range** (spatial structure)
- Different **sill** (overall variability); **nugget** (uncertainty at sample point)

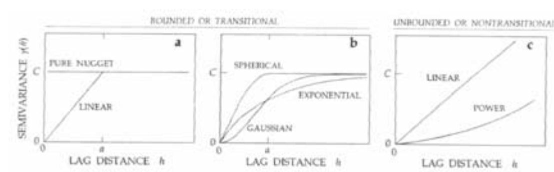
Others are possible but much more complicated to estimate.

Sample size

- In general, 30 or more points are needed to generate a reasonable sample variogram.
- The most important part of a variogram is its shape near the origin as closest points are given more weight in the interpolation process.



Variogram models



Theoretical models of seven different kinds of semivariograms.

Variogram models

Equations for the seven semivariogram models

Model	Equation	Consideration
Bounded or Transitional		
Pure nugget	$\gamma(h) = \begin{cases} 0 \\ C \end{cases}$	$h = 0$; $h > 0$
Linear	$\gamma(h) = \begin{cases} Ch/a \\ C \end{cases}$	$0 \le h \le a$; $h > a$
Spherical	$\gamma(h) = \begin{cases} C\left[\frac{3h}{2a} - \frac{1}{2}\left(\frac{h}{a}\right)^3\right] \\ C \end{cases}$	$0 \le h \le a$; $h > a$
Exponential	$\gamma(h) = C[1 - \exp(-h/a)]$	$h \ge 0$
Gaussian	$\gamma(h) = C\{1 - \exp[-(h/a)^2]\}$	$h \ge 0$
Unbounded or Nontransitional		
Linear	$\gamma(h) = mh$	$h \ge 0$
Power	$\gamma(h) = mh^{\beta}$	$h \ge 0; 1 < \beta < 2$

Co-Kriging : Prerequisites

1. Two point data sets, usually with some observations co-located:
   (a) the **target variable** z at locations ~$x1$ . . .~xn_z
   (b) the **co-variable** w at locations ~$y1$ . . .~yn_w
2. **Spatial structure** in **both variables separately** (i.e. non-nugget **variograms**);
3. **Spatial structure between the variables** (**cross-variograms**); this can be either a positive or negative spatial correlation;
4. Certain restrictions on the joint spatial structure.

The Co-kriging predictor

Prediction of the target variable at an unknown point x_0 is computed as the **sum of two weighted averages**:

1. one of the n_z sample values of the **target variable** z; and
2. one of the n_w sample values of the **co-variable** w

$$z(\vec{x}_0) = \sum_{i=1}^{n_z} \lambda_i z(\vec{x}_i) + \sum_{j=1}^{n_w} \mu_j w(\vec{y}_j)$$

We want to find the weights λ (for the target variable) and μ (for the co-variable) which minimize the prediction variance.

The Co-kriging variance

Suppose there are V variables (target and co-variables), which are indexed from $1 \ldots l$; each has n_l observations.

Variable u is the target, one of the V. We want to minimize the variance.

$$\sigma_u^2 = \sum_{l=1}^{V} \sum_{j=1}^{n_l} \lambda_{jl} \gamma_{ul}(x_j, x_0) + \psi_u$$

So the semivariance of all observations of all variables with the point to be estimated is minimized.

Note there is one of these equations for each variable.

The Co-Kriging system

Solve: $\mathbf{A_C} \lambda_\mathbf{C} = \mathbf{b_C}$, where $\mathbf{A_C}$ is built up from the direct and cross-semivariances of the sample points (both target and co-variable):

$$\mathbf{A}_\mathrm{C} = \begin{bmatrix} \Gamma_{zz} & \Gamma_{zw} & \mathbf{1} & \mathbf{0} \\ \Gamma_{wz} & \Gamma_{ww} & \mathbf{0} & \mathbf{1} \\ \mathbf{1}^\mathrm{T} & \mathbf{0}^\mathrm{T} & 0 & 0 \\ \mathbf{0}^\mathrm{T} & \mathbf{1}^\mathrm{T} & 0 & 0 \end{bmatrix}$$

- The vectors of 1's and 0's control which of the semi-variances are included in each equation.
- The Γ are the matrices of semi-variances (next slide)

Example of a cross-variable matrix

$$\Gamma_{zw} = \begin{bmatrix} \gamma_{zw}(\vec{x}_1, \vec{y}_1) & \gamma_{zw}(\vec{x}_1, \vec{y}_2) & \cdots & \gamma_{zw}(\vec{x}_1, \vec{y}_{n_w}) \\ \gamma_{zw}(\vec{x}_2, \vec{y}_1) & \gamma_{zw}(\vec{x}_2, \vec{y}_2) & \cdots & \gamma_{zw}(\vec{x}_2, \vec{y}_{n_w} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{zw}(\vec{x}_{n_z}, \vec{y}_1) & \gamma_{zw}(\vec{x}_{n_z}, \vec{y}_2) & \cdots & \gamma_{zw}(\vec{x}_{n_z}, \vec{y}_{n_w}) \end{bmatrix}$$

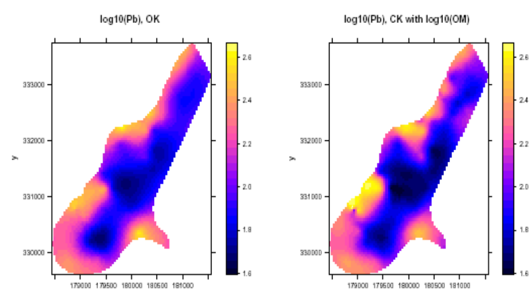
The Co-Kriging system

$$\lambda_\mathrm{C} = \begin{bmatrix} \lambda_1 \\ \cdots \\ \lambda_{n_z} \\ \mu_1 \\ \cdots \\ \mu_{n_w} \\ \psi_z \\ \psi_w \end{bmatrix} \qquad \mathbf{b}_\mathrm{C} = \begin{bmatrix} \gamma_{zz}(\vec{x}_1, \vec{x}_0) \\ \vdots \\ \gamma_{zz}(\vec{x}_{n_z}, \vec{x}_0) \\ \gamma_{wz}(\vec{y}_1, \vec{x}_0) \\ \vdots \\ \gamma_{wz}(\vec{y}_{n_w}, \vec{x}_0) \\ 1 \\ 0 \end{bmatrix}$$

The **cokriging prediction variance** is then:

$$\hat{\sigma}_z^2(\vec{x}_0) = \mathbf{b}_\mathrm{C}^\mathrm{T} \lambda_\mathrm{C}$$

Difference between OK and CK predictions



Exercises

- Computer program: Geostatistics +