2016-17 AY Assessment Report
Bachelors in Arts in Mathematics
Department of Mathematics. Fresno State

1. **Learning outcomes assessed this year**

Direct Assessment.

- We assessed one of our core courses, MATH 152 (Linear Algebra), in both semesters of the 2016-17 AY.

  We focused on Goals A, B, and C. More specifically, we assessed SLOs A1, B1, and C1. Next is the description of these goals and SLOs:

  Goal A. Provide students with conceptual background knowledge in the core areas of mathematics.

  A1. Students will understand and use the definitions and basic properties of fundamental concepts in algebra and analysis, such as function, derivative, integral, matrix, group.

  Goal B. Teach students to read, understand, and write rigorous mathematical proofs.

  B1. Students will be familiar with common notations and proof techniques.

  Goal C. Provide students with opportunities to apply mathematical knowledge to solve theoretical and practical problems.

  C1. Students will use their knowledge of calculus and linear algebra to solve practical application problems.

- By request of its coordinator, we assessed MATH 6 (Precalculus) in Spring 2017. This is a service course taken by, mostly, STEM majors that are not ready for Calculus when they get to Fresno State. In particular, MATH 6 is not a credit-bearing class for Math majors. Hence, it falls outside of our B.A. program, and thus our program SLOs do not apply to it. Because of this, we slightly modified our program SLOs to the following ones, just for the assessment of this class.

  A1'. Students will understand and use the definitions and basic properties of fundamental concepts in pre-calculus, such as functions and their basic properties and graphs, rational functions, exponentials, logarithms, and concepts in trigonometry.

  C1'. Students will use their knowledge to solve practical application problems.

  D1'. Students will be able to explain their solutions in writing.

Indirect Assessment.

We surveyed graduating students' perceptions of our program, and department.

2. **What instruments did you use to assess them?**

- **A1.** A total of four embedded questions on the final exams in MATH 152 in the 2016-17 AY.
- **B1.** A total of four embedded questions on the final exams in MATH 152 in the 2016-17 AY.
- **C1.** A total of four embedded questions on the final exams in MATH 152 in the 2016-17 AY.
- **A1'.** One embedded question on the final exam in MATH 6 in Spring 2017.
- **C1'.** One embedded question on the final exam in MATH 6 in Spring 2017.
- **D1'.** One embedded question on the final exam in MATH 6 in Spring 2017.

Exit interviews and online exit surveys of students in the graduating class were used to capture students' feelings and thoughts about our program and department.

3. **What did you discover from these data?**

MATH 152:
We assessed three sections in the 2016-17 AY.
Section 1. **A total of 11 students took this exam. 4 students out of the 11 who took the final exam passed the exam (3 Cs and 1 A).** The mean for the final was a 54% (an F).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were A1: 81% (a B), B1: 37% (an F), C1: 48% (an F).

Section 2. **A total of 30 students took this exam. 21 students out of the 30 who took the final exam passed the exam (7 Cs, 12 B, 2 A).** The mean for the final was a 66% (a C).
The means of the three embedded questions, and the letter grade (according to the instructor's grading scale), were A1: 52% (a D), B1: 10% (an F), C1: 89% (an A).

The assessment of Sections 1 and 2 was done after finals were given and graded; we acted retroactively during the beginning of the Spring semester. No assessment for these sections was performed during the Fall. Hence, for Section 3 (it ran during the Spring), we have more details on how the data breaks. Each question was measured for two different SLOs.

Section 3. **A total of 33 students (out of 36 on the roster) took this exam. 22 students out of the 33 who took the final exam passed the class with a C or better.** The course grade distribution was (according to the instructor's grading scale): 6A, 7B, 9C, 10D, 4F and 2W.

MATH 6:

The objective of the assessment this course was to compare the performance between students enrolled in a section that was taught in a different way and students in 'regular' sections. Hence, there were no expectations about how students should perform in each question assessed.

The section that is being compared to all the others was special because the instructor in that section was:

- using a novel pedagogical method of instruction referred to as 'flipping' their classroom. The lectures were recorded and students watched them outside of class. In-class activities focused on group problem solving and presentation of solutions.
- requiring students to attend supplementary instruction/problem sessions by including participation in course credit.

All these extra ways to support students were optional in the other sections, or just non-applicable.

Question 1. SLO A1'. The average score for this problem in the control section was 4.63 out of 15, the average score for this problem in all the other sections combined was 4.93 out of 15.

Question 2. SLO C1'. The average score for this problem in the control section was 8.08 out of 15, the average score for this problem in all the other sections combined was 7.09 out of 15.

Question 3. SLO D1'. The average score for this problem in the control section was 4.17 out of 10, the average score for this problem in all the other sections combined was 5.50 out of 10.

Summarizing, we identify the following issues/remarkable facts.

- *MATH 152:* Weak proof skills in students throughout. Also, given that none of the instructors succeeded in getting their students to score high in both the questions measuring applied and the more theoretical questions, we believe that the instructors are unable to cover effectively both types of material.

We analyzed the data, running a Poisson regression model where the response variable is *FINAL* and the predictor variables are *A1*, *B1*, *C1*, and *Instructor*, and the results showed that all of the instructors had difficulty covering both abstract and applied material, as measured by final exam achievement.
We summarize this analysis in the following table:

	Estimate	Std. Error	p-value
Intercept	56.827	6.993	0.000**
A1	12.364	4.968	0.015**
B1	44.576	8.216	0.000**
C1	52.248	8.643	0.000**
Instructor 2	-13.750	19.377	0.480
Instructor 3	21.330	6.855	0.003**

** indicates significant at 5% level.

As expected and indicated by the p-values in the table, the SLO outcomes *A1*, *B1*, *C1* significantly affect the final exam score. The variable *Instructor* is categorical and the first instructor (Instructor 1) was taken as the baseline (reference) category. The result indicates that when comparing Instructor 1 and Instructor 2, *Instructor* has no significant effect on the final exam (the Estimate falls within the Standard Error). On the other hand, Instructor 3 had significant positive effect on the final exam (~ 21 points for a ~ 7 Standard Error). This may be due, in part, to extra credit points given by this instructor in the exam.

Complementing this analysis on final exam scores, students also identified the difficulty of proofs in Math 152 in their exit interviews and surveys. Students felt that it was unfair that we required proofs in MATH 152 while our introduction to proofs class (MATH 111) was not a pre-requisite.
Finally, Instructor 3 said that, according to their appreciation, there was no clear difference between the performance of Math Majors and non-Math Majors in the more theoretical questions. The same situation was observed when comparing students who took MATH 111 and those who did not.

- *MATH 6:* We computed the means in each question assessed, for the control section and for all the others combined. The results of this are summarized next.

	Q1 (max=10 pts)	Q2 (max=15 pts)	Q3 (max=15 pts)
All sections except control	5.50	4.93	7.09
Control section	4.17	4.63	8.08

Hence, we did not find any significant difference between the 'control section' and all the others.

To gauge how students in the 'control section' responded to the redesigned activities, which were mainly the lecture videos, five surveys were given to the students. The surveys were developed and administered with the help/supervision of our colleague, Dr. Jenna Tague (not the instructor of this section), based on her previous research on the effectiveness of flipped classrooms (Tague & Czocher, 2016). The surveys were designed to determine if the students were watching the videos, if the students saw how the videos and in-class activities were related, and then various technical aspects of the videos.

1. I watched the lecture videos.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
2. The pre-class videos were related to the in-class activities.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
3. Having videos to watch and think about before class was helpful to my learning.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
4. The online lecture videos were easily accessible.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
5. The quality of the lecture videos met my expectations.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
6. More math lectures should make lecture videos available online.
   Strongly Agree　　　Strongly Disagree
   1　2　3　4　5
7. How often did you attend lecture?
   Always　　　Sometimes　　　Never
   1　2　3　4　5

The survey results indicate that the students self-assessed as watching the videos consistently over the course of the semester. The last survey, which was given in the last full week of the semester, was the only survey where less than 90% of the students rated their watching as neutral or below. The most remarkable finding, however, was that over 86% of the students in each survey strongly agreed that the out-of-class materials were related to the in-class activities. One of the weaknesses of the flipped classroom is that it can leave students not understanding the goals of the course, and feeling like the different parts are unrelated (Bowers & Zazkis, 2012). That feeling of disconnectedness was not present in the data collected from the students. The students also overwhelmingly and consistently (strongly agreed, agreed) found the redesigned course materials helpful, accessible, and of a good quality. Lastly, it should be noted that the number of students surveyed remains consistent, which is unusual in an entry-level mathematics course. The reason for the consistent levels of students surveyed was because the attendance in the course did not waiver over the course of the semester, which again, is remarkable in an entry-level mathematics course.

The only challenge that appears in the data in terms of watching and accessibility was that the videos were hosted through Fresno State's servers. For a few of the students, they could only watch the videos when they were on-campus. Because many of the Fresno State students are commuters and work full time jobs, this presented a struggle for those students. In future semesters, we are checking other hosting options to try to alleviate this difficulty for those few students.

Survey 1 (N=24)

	Watched	Related	Helpful	Accessible	Quality	More	Attend
SA	92%	88%	79%	83%	79%	75%	83%
A	4%	8%	17%	13%	13%	17%	13%
N	4%	4%	0%	0%	0%	0%	0%
D	0%	0%	0%	0%	0%	4%	0%
SD	0%	0%	0%	0%	0%	0%	0%

Survey 2 (N=22)

	Watched	Related	Helpful	Accessible	Quality	More	Attend
SA	86%	91%	77%	64%	73%	82%	95%
A	14%	5%	5%	9%	5%	9%	0%
N	0%	0%	14%	14%	18%	0%	0%
D	5%	0%	0%	5%	0%	0%	0%
SD	0%	0%	0%	5%	0%	5%	0%

Survey 3 (N=22)

	Watched	Related	Helpful	Accessible	Quality	More	Attend
SA	95%	86%	82%	68%	68%	86%	91%
A	0%	5%	9%	18%	32%	9%	9%
N	0%	5%	5%	14%	0%	0%	0%
D	5%	0%	5%	0%	0%	0%	0%
SD	0%	0%	0%	0%	0%	5%	0%

Survey 4 (N=22)

	Watched	Related	Helpful	Accessible	Quality	More	Attend
SA	95%	90%	81%	67%	57%	81%	90%
A	0%	0%	10%	24%	33%	14%	5%
N	5%	5%	5%	5%	5%	0%	0%
D	0%	5%	5%	5%	5%	5%	5%
SD	0%	0%	0%	0%	0%	0%	0%

Survey 5 (N=22)

	Watched	Related	Helpful	Accessible	Quality	More	Attend
SA	63%	89%	89%	56%	72%	89%	94%
A	11%	11%	11%	22%	17%	11%	6%
N	0%	0%	0%	17%	11%	0%	0%
D	5%	0%	0%	6%	0%	0%	0%
SD	5%	0%	0%	0%	0%	0%	0%

We can see that, although students in the control section are positive about the way the class was taught, this positive disposition did not translate in a better performance in the questions assessed. Hence, the assessment committee will not make any recommendations regarding this class until more data is collected.

The following findings come exclusively from the student exit surveys/interviews.

- The exits interviews/surveys are fairly positive. Students feel their experience in the math program is rough and challenging, but good. The places where we have some room for improvement, according to these interviews/surveys, are:
  (a) *Upper division courses, mostly MATH 111 and MATH 171:* Students, in their exit surveys and interviews, complained that the difficulty of certain courses depended heavily on the instructor. Complaints were varied in type; some of them talked about students avoiding certain instructors, and some others wished a course that was a pre-requisite for some other had been more demanding so they would have

been ready for the sequel course.
More specifically, most students felt that MATH 111 did not do enough to 'prepare' students for MATH 171.

(b) *LaTeX and Mathematical Software:* Students did not complain about LaTeX being required in certain classes, but about them not receiving enough training, and also about the fact that LaTeX was not required consistently in our courses. Training on how to use mathematical software, such as Mathematica, Maple, GeoGebra, etc. would also be desirable.

(c) *Advising:* Students agreed that they did not feel the need of getting advising; few got clear benefits of meeting their advisor (besides lifting holds). Many of them had bad experiences (adviser showing not much interest in helping out or giving incorrect advising, and also ARC not working as well as we would expect).
Several students did not know that they could switch advisers to find somebody who matches their needs better.

(d) *Schedule:* Students seemed to prefer classes taught four days a week, instead of in two blocks of two hours. Also, they suggested that our earliest class should be at 9:00AM.
They also suggested that Math Club meetings were held on Thursdays, as very few MATH upper division classes run on Fridays and thus students are not on campus for a Friday meeting. We remark on this not only for our Math Club officers to take note, but also for the scheduling of seminars, and other activities that our department wants students to get involved in.

4. **What changes did you make as a result of the findings?**

Our department does not currently have a plan for 'closing the loop.' Hence, actions will be taken after our department discusses our recommendations, and only if the faculty in our department agrees with our assessment and recommendations.

From the summary at the end of the previous item, we identified three issues that we believe should be addressed by our department:

(i) *The applied-theoretical dichotomy present in our MATH 152 course.* We suggest the creation of a new course, in which Math majors would get a more abstract perspective on the topics in MATH 152; this would be a proof-based course and it would have MATH 111 as a pre-requisite, and the current MATH 152 would be taught keeping in mind non-Math Majors (mostly Engineering students) and thus it would not require students to dwell in the more theoretical aspects of Linear Algebra.

(ii) *Technology.* We should have ways to support students in the learning of mathematical software, besides what little can be done in our courses. Adding this to MATH 111 would be a solution, or we could also create a 1-unit course on technology for math majors. We need to expose our students to LaTeX, Mathematica, GeoGebra, Excel,

Python, R, etc.
Another solution could be to have workshops on technology at the beginning of every semester and/or to have some kind of 'support' that they can access during the semester.

(iii) *Advising.* We should make an effort to reach out to our students so they are more enthusiastic about getting advising from us. We suggest that we create documentation that is useful to both advisors and students at the time of performing advising.
Possible documents to create are: (1) more detailed road maps (including 'desirable' prerequisites), (2) special road maps for transfer students to share during Dog Days, (3) an 'advising template' that could help faculty to give better advising (some kind of advising training could work as well), etc.
In this area, our department has recently made some changes to create a more welcoming, and inviting, environment for students; we are requiring transfer students to attend the Math Welcome Event in the Spring before they can register in the Fall. This is in addition to the Math Welcome Event in the Fall that is meant for incoming freshmen. It is likely the students graduating this year did not experience that event, so it is unclear whether these events will help the students gain a better understanding of advising.

Finally, in the 2015-16 Assessment Report, there were two items that were mentioned as changes that would need to occur during this AY. The status of these changes follows:

(a) *MATH 111.* As mentioned in last year's report, our department created a committee to study how to improve this course. The committee has not finished its work, and so far the only decision made by it is that MATH 111 would go from being a 3-unit course to a 4-unit course.

(b) *MATH 75.* In last year's report it is said that

> "A new, more relevant to our Math 75, Calculus Readiness Test is currently being written. The comparison of student performance in Math 75 versus CRT scores will help determine appropriate cut-off scores."

In AY 2016-17, the Chancellor's Office requested all CSU mathematics departments explore the use of ALEKS PPL assessments for mathematics placement into calculus and other mathematics courses after successful implementation at several CSU campuses. The department has focused on this as a CRT replacement rather than rewrite a new placement. In August 2017, the ALEKS PPL was piloted for incoming international students. In AY 2017-18, our department will discuss whether to adopt the use of ALEKS PPL to replace the CRT and other assessments. There will be limited pilots in precalculus courses in Fall 2017.

5. **What assessment activities will you be conducting in the 2017-18 academic year?**

   (a) In the next AY, we will change the way we do assessment by moving from assessing courses to assessing goals. We will start this by looking at

   > **Goal B.** Teach students to read, understand, and write rigorous mathematical proofs.

   In order to assess this goal we have chosen three courses in which we will look at the expected evolution in students' achievement of goal B. Next are the courses we will embed questions in, together with their expectations for SLOs related to goal B.

   - MATH 111 = I
   - MATH 165 = R
   - MATH 171 = M

   (b) Use Tableau to study how well MATH 111 prepares students for proof-based upper division courses.

   (c) Re-write our SOAP so our department can effectively assess our new BS in Mathematics program, which will replace our current BA starting in Fall 2018. This new program will feature several new courses, and five different options.
   An important part of the new SOAP will be how we will assess our options, and what courses (or goals) will be evaluated often to create reports on them every five years.

   (d) Assess the usage of technology in our courses.

   (e) As mentioned before, our department does not have a plan to 'close the loop'. We believe that without action, assessment reports like this one are useless. Hence, during the 2017-18 AY we will work on a plan to close the loop; we expect this project to involve all our department.

   The reason that our assessment plan for next year is different from that in our current SOAP is because of our impending change from BA to BS, and because we wanted to try something different for assessment (Goals instead of Courses), so we can see what type of assessment our department should continue doing when we start with our BS program.

6. **What progress have you made on items from your last program review action plan?**

   During the 2016-17 AY, we had a site visit to review our programs. The visit occurred on Sept. 28th and 29th, 2016. The review panel consisted of Prof. Kim Morin, Theatre Arts, CSU Fresno, Dr. Saeed Attar, Professor of Chemistry, Director of Honors College, CSU Fresno, and Dr. Ivona Grzegorczyk, Professor and Chair, Department of Mathematics, California State University, Channel Islands.

   The panel delivered the following recommendations. We have also included corrections to a few significant inaccurate statements in the report.

A. Curriculum Improvements and Vision for the Future.

Recommendation 1. The reviewers strongly support the proposal for a BS degree with a stronger curriculum in the areas of research and applied mathematics, and urge the administration to help the department put the proposal on the 'fast track' for approvals.
Response. Our BS program proposal will be submitted in early Fall 2017. Hence, we should start offering a BS in Mathematics starting in Fall 2018.

Recommendation 2. The department should consider a BS degree with options in pure mathematics, applied mathematics, statistics, and secondary mathematics teaching.
Response. All those options are considered in our current BS plan.

Recommendation 3. The program should update the curriculum flow chart to help students navigate through the major requirements.
Response. This is one of the recommendations we are giving in this report (see item 4(iii)).

Recommendation 4. Faculty should discuss a long-term vision for the department.
Response. We have had a couple of retreats, in Fall 2015 and fall 2016, to discuss this. Besides the implementation of a BS in Mathematics, which includes a 'blended' program for students who want to teach at the High School level, no further actions have been taken. We have discussed the creation of a 4+1 program in the near future.

B. Supporting Faculty Research and Workload Issues.

Recommendation (Administration and the Department). Identify sources for long term funding so the program can offer release time or summer stipends to faculty engaging in research and grant-writing activities.
Response. No progress.

C. Departmental Budget.

Recommendation. Identify College and University funds to be included in the departmental funding base for faculty scholarly activities and curriculum coordination.
Response. No progress.

D. Improving technology use in mathematics courses.

Recommendation. Rethink delivery of the calculus, statistics, and upper division courses to include updated use of technology and current mathematical software.

Response. No progress. Any advances on this regard have been made by individuals; our department does not have a plan for this contingency.

E. Supporting Undergraduate and Graduate Student Research.

Recommendation 1. Rethink the ways to involve undergraduate and graduate students into original research rewarding supervising faculty with adequate work load.
Response. No progress on the 'reward' end. Our department's student research committee has created a simple system to identify students who want to engage in research and to match them with faculty willing, and able, to mentor them.
Also, students in our upcoming BS program will have to have a culminating experience, which will involve seminars and, in certain options, a research senior project.

Recommendation 2. Create funding for the department to support small courses for faculty student research projects.
Response. No progress

Recommendation 3. Explore the possibility to offer research courses, where full course load is given to faculty for working with small groups of students.
Response. No progress

F. Facilities.

Recommendation (Administration) 1. Try to locate all faculty and graduate student offices in closer proximity to the department.
Response. Our Dean has provided three additional offices for our part time faculty to share. Even after this, our need for part time faculty and TA office space remains severe.

Recommendation (Administration) 2. Provide additional space that is equipped appropriately for best practices in teaching mathematics that will facilitate faculty/student collaboration and research activities.
Response. No progress.

Recommendation (Administration) 3. Investigate the use of laptops to meet the computing needs of the faculty and students.
Response. All full-time faculty have laptops provided by our college. There has been no progress regarding part-time faculty and students (including our TAs).

G. Involving Lecturers in Departmental Activities.

Recommendation 1. Include lecturers in programmatic issues relevant to the courses they teach (especially in committees on instruction and curriculum related to their teaching assignments).
Response. No progress.

Recommendation 2. Allocate an additional appropriate space for the program designated to faculty-student collaborations and projects.
Response. No progress.

H. Assessment and Student Learning Outcomes

Recommendation. The Student Learning Outcomes should include familiarity with current technology accepted by the mathematical community.
Response. No progress. Any advances on this regard have been made by individuals; our department does not have a plan for this contingency.

MATH 6, Assessment Questions and Rubric. 2016-17 AY

The questions, and their rubrics, used in the common final exam of MATH 6 follow.

Question 1. SLO A1'. Solve for x: $\log_2(3x-5) = 3 + \log_2(6x+5)$.

Solution.

- State that the conditions $3x-5>0$ and $6x+5>0$ are necessary, or use these inequalities to check for extraneous solutions at the end. **3 points**.
- Re-write equation and combine logarithms. **5 points**.

$$\begin{aligned} \log_2(3x-5) &= 3 + \log_2(6x+5) \\ \log_2(3x-5) - \log_2(6x+5) &= 3 \\ \log_2\left(\frac{3x-5}{6x+5}\right) &= 3 \end{aligned}$$

- Transform the equation into an equation without logarithms. **3 points**.

$$\log_2\left(\frac{3x-5}{6x+5}\right) = 3 \implies \frac{3x-5}{6x+5} = 2^3 = 8$$

- Solve the equation. **4 points**.

$$\begin{aligned} \frac{3x-5}{6x+5} &= 8 \\ 3x-5 &= 8(6x+5) \\ 3x-5 &= 48x+40 \\ -45x &= 45 \\ x &= -1 \end{aligned}$$

- Use inequalities found in first step to check the solution obtained.

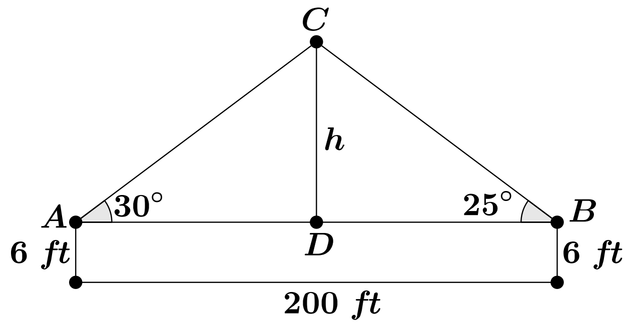
$$3(-1)-5 = -8 \not> 0$$

Hence, no solutions.

Question 2. SLO C1'. Andrew and Bart (who are both 6 feet tall) stand 200 feet apart with a vertical flagpole somewhere between them. Andrew's angle of elevation to the top of the flagpole is 30° while Bart's angle of elevation to the top of the flagpole is 25°. Find the height of the flagpole. Round your answer to the nearest foot.

Solution.

- Draw a picture capturing all the date given. **4 points**.



- Find the measure of $\angle C$ In $\triangle ABC$. **3 points**.

$$m(\angle C) = 180° - 30° - 25° = 125°$$

- Use the theorem of sines in $\triangle ABC$ to find the length, b, of the side opposite to B . **3 points**.

$$\frac{b}{\sin 25°} = \frac{200}{\sin 125°}$$
$$b = \frac{200 \cdot \sin 25°}{\sin 125°}$$
$$b = 103.18\ ft$$

- Use trigonometry in $\triangle ADC$ to find the value of h. **3 points**.

$$\sin 30° = \frac{h}{103.18}$$
$$h = \sin 30° \cdot 103.18$$
$$h = 51.59\ ft$$

- Compute the height of the flagpole. **2 points**.

$$51.59 + 6 = 57.59 \sim 58\ ft$$

Question 3. SLO D1'. Evaluate $\tan^{-1}(-\sqrt{3})$ without using a calculator. Show your work.

Solution.

- Re-write the quantity we want to find as an equation using tangents, and mention restrictions for possible solutions of resulting equation. **3 points**.

$$\tan^{-1}(-\sqrt{3}) = \theta \iff -\sqrt{3} = \tan\theta$$

where $-90° < \theta < 90°$

- Identify the reference angle for the equation obtained. **4 points**.

$$\tan 60^\circ = \sqrt{3} \quad \Longrightarrow \quad \text{reference angle is } 60^\circ$$

- Find a solution for θ using the reference angle and when the sign of the tangent function is negative. **3 points**.
  Since $\tan\theta < 0$ and $-90^\circ < \theta < 90^\circ$, then θ is in the fourth quadrant. Hence,

$$\theta = -60^\circ = -\frac{\pi}{3}$$

  It follows that $\tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$.

MATH 152, Assessment Questions and Rubric. 2016-17 AY

The questions used by Instructors 1 and 2 follow.

SLO A1.

1. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -2 & 4 & 5 & 3 \\ 3 & -6 & -5 & -2 \end{bmatrix}$

   (a) Find a basis for $N(A)$.
   (b) Find a basis for $row(A)$.
   (c) Find a basis for $col(A)$.

2. Let $\mathbf{A} = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$.

   (a) Determine if $\mathbf{w}$ is in $Col\mathbf{A}$.
   (b) Is $\mathbf{w}$ in $Nul\mathbf{A}$? Justify.

SLO B1.

1. (a) Suppose that $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set in $\mathbb{R}^3$ and that A is a invertible 3×3 matrix. Prove that $T = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3\}$ is linearly independent.
   (b) Does $T = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3\}$ span $\mathbb{R}^3$? Why or why not?

2. A *Householder matrix*, or an elementary reflector, has the form $Q = I - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{u}$ is a unit vector. Show that Q is an orthogonal matrix.
   (Elementary reflectors are used in computer programs to produce a QR factorization of a matrix A. If A has linearly independent columns, then left-multiplication by a sequence of elementary reflectors can produce an upper triangular matrix.)

SLO C1.

1. Let S be the subspace of of $M_{2\times 2}$ consisting of symmetric 2×2 matrices. Define the linear transformation $T : S \to P_2$ by

$$T \begin{bmatrix} a & b \\ b & c \end{bmatrix} = c + (a+b)x^2$$

   Note: You do NOT need to verify that T is a linear transformation.
   (a) Find the kernel of T. Is T injective?
   (b) Find the range of T. Is T surjective?

2. Use determinants to find out if the matrix is invertible: $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

The questions, and their rubrics, used by Instructor 3 follow.

Question 1. SLOs A1 and C1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ s \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

1. Use concepts learned in this class, to find the volume of the parallelepiped spanned by the three vectors.

2. For what value(s) of s is the volume 0? Interpret this geometrically.

Solution. This problem can be solved in several ways. In this rubric we discuss the two solutions students are most likely to come up with. **10 points total**.

Short way: Here we use determinants.

- Form a matrix whose rows (or columns) are the given three vectors. **3 points**.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ s & 0 & 3 \end{bmatrix}$$

- Compute the determinant of A. **3 points**.

$$\begin{aligned} det(A) &= \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ s & 0 & 3 \end{vmatrix} \\ &= (1 \cdot 1 \cdot 3 + (-2) \cdot 1 \cdot s + 1 \cdot 2 \cdot 0) - (1 \cdot 1 \cdot s + (-2) \cdot 2 \cdot 3 + 1 \cdot 0 \cdot 1) \\ &= (3 - 2s) - (s - 12) \\ &= 15 - 3s \end{aligned}$$

- Use that $Volume = |det(A)|$. **2 points**.
  We get that the volume of the parallelepiped is $V = |15 - 3s|$.

- Set $det(A) = 0$ and solve for s. **1 point**.
  Setting the volume equal to zero, we get $15 - 3s = 0$, and thus $s = 5$.

- Interpret geometrically and provide an answer. **1 point**.
  For $s = 5$ the vectors are linearly dependent, and so the volume of the parallelepiped created by them is equal to zero.

Long way: This solution uses the Gram-Schmidt orthogonalization process.

- Find the area of the base of the parallelepiped. **3 points**.
  Consider, say $\mathbf{v}_2$ and $\mathbf{v}_3$ (they do not have s, so this would be easier than picking another pair of vectors), and find an orthogonal basis of the space spanned by these vectors. We follow the steps

  1. $\mathbf{v}_2 - proj_{\mathbf{v}_3}\mathbf{v}_2 = \mathbf{v}_2'$. **1 point**.

$$\begin{aligned}
\mathbf{v}_2' &= \mathbf{v}_2 - proj_{\mathbf{v}_3}\mathbf{v}_2 \\
&= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{11} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\
&= \frac{1}{11} \begin{bmatrix} -21 \\ 12 \\ 3 \end{bmatrix}
\end{aligned}$$

  2. Compute the length of $\mathbf{v}_2'$. **1 point**.

$$\begin{aligned}
||\mathbf{v}_2'|| &= \left\| \frac{1}{11} \begin{bmatrix} -21 \\ 12 \\ 3 \end{bmatrix} \right\| \\
&= \frac{1}{11}\sqrt{(21)^2 + (12)^2 + (3)^2} \\
&= \frac{3}{11}\sqrt{66}
\end{aligned}$$

3. Compute the length of $\mathbf{v}_3$ and get $Area = \|\mathbf{v}_2'\| \cdot \|\mathbf{v}_3\|$. **1 point**.

$$\begin{aligned} ||\mathbf{v}_3|| &= \left\| \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\| \\ &= \sqrt{1^2 + 1^2 + 3^2} \\ &= \sqrt{11} \end{aligned}$$

thus

$$\begin{aligned} Area &= \|\mathbf{v}_2'\| \cdot \|\mathbf{v}_3\| \\ &= \frac{3}{11}\sqrt{66} \cdot \sqrt{11} \\ &= 3\sqrt{6} \end{aligned}$$

- Get the height vector of the parallelepiped. **3 points**.

$$\begin{aligned} H &= \mathbf{v}_1 - proj_{\mathbf{v}_2'}\mathbf{v}_1 - proj_{\mathbf{v}_3}\mathbf{v}_1 \\ &= \begin{bmatrix} 1 \\ 2 \\ s \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ s \end{bmatrix} \cdot \frac{1}{11}\begin{bmatrix} -21 \\ 12 \\ 3 \end{bmatrix}}{\frac{3^2}{11^2}(66)} \frac{1}{11}\begin{bmatrix} -21 \\ 12 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 2 \\ s \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\ &= \frac{5-s}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \end{aligned}$$

- Compute $||H||$. **1 point**.

$$||H|| = \left\| \frac{5-s}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\| = \frac{|5-s|}{6}\sqrt{1^2 + 2^2 + (-1)^2} = \frac{|5-s|}{\sqrt{6}}$$

- Compute the volume of the parallelepiped. **1 point**.

$$Volume = Area||H|| = 3\sqrt{6} \cdot \frac{|5-s|}{\sqrt{6}} = 3|5-s|$$

- Set the volume equal to zero and solve for s. **1 point**.

$$0 = 3|5-s| \implies s = 5$$

- Interpret geometrically and provide an answer. **1 point**.
  For $s = 5$ the vectors are linearly dependent, and so the volume of the parallelepiped created by them is equal to zero.

Question 2. SLOs A1 and B1. Consider the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and the function $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{u}$ for all $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{x} \cdot \mathbf{u}$ is the dot product in $\mathbb{R}^3$. Show that T is a linear transformation and find a basis for its kernel. Is T one-to-one? Is T onto?
Solution. 10 points total.

- Know what a linear transformation is to be able to then prove that the given function is linear. **1 point**.
  We need to check that $T : \mathbb{R}^3 \to \mathbb{R}$, given by $T(\mathbf{x}) = \mathbf{x} \cdot \mathbf{u}$ satisfies.

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}), \text{ for all } \alpha, \beta \in \mathbb{R}, \text{ and, for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$$

- Check that T is linear. **3 points**.
  In order to do this we use that $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to re-define T as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

  Hence,

$$\begin{aligned}
T(\alpha\mathbf{x} + \beta\mathbf{y}) &= T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{bmatrix}\right) \\
&= (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) + 3(\alpha x_3 + \beta y_3) \\
&= (\alpha x_1 + 2\alpha x_2 + 3\alpha x_3) + (\beta y_1 + 2\beta y_2 + 3\beta y_3) \\
&= \alpha(x_1 + 2x_2 + 3x_3) + \beta(y_1 + 2y_2 + 3y_3) \\
&= \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)
\end{aligned}$$

- Express $N(T)$ in set notation. **1 point**.

$$N(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3;\ T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3;\ x_1 + 2x_2 + 3x_3 = 0 \right\}$$

- Find a basis for $N(T)$. **3 points**.

$$\begin{aligned} N(T) &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3;\ x_1 + 2x_2 + 3x_3 = 0 \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3;\ x_1 = -2x_2 - 3x_3 \right\} \\ &= \left\{ \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix};\ x_2, x_3 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix};\ x_2, x_3 \in \mathbb{R} \right\} \\ &= \textbf{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

  This is a basis because the vectors are clearly linearly independent.

- Provide details justifying whether T is 1-1. **1 point**.
  Since the null-space is not trivial then T is not injective.

- Provide details justifying whether T is onto. **1 point**.
  Since the dimension of the null-space is two and the domain has dimension three, then the dimension of the range is one, which coincides with the dimension of the co-domain. Hence, T is onto.

<u>Question 3. SLOs B1 and C1.</u> Let $C = B^{-1}AB$. Show that if $\mathbf{v}$ is an eigenvector of C corresponding to the eigenvalue λ, then $B\mathbf{v}$ is an eigenvector of A corresponding to λ.
Solution. 10 points total.

- Know that they should pick a non-zero eigenvector associated to the eigenvalue λ. **1 point**.

- Know that $C(\mathbf{v}) = \lambda\mathbf{v}$. **1 point**.

- Setup $(B^{-1}AB)(\mathbf{v}) = \lambda\mathbf{v}$ and get $B(B^{-1}AB)(\mathbf{v}) = B(\lambda\mathbf{v})$. **2 points**.
- Know that $BB^{-1} = I$. Then get from $B(B^{-1}AB)(\mathbf{v}) = B(\lambda\mathbf{v})$ that $(AB)(\mathbf{v}) = B(\lambda\mathbf{v})$. **2 points**.
- Know that $B(\lambda\mathbf{v}) = \lambda B(\mathbf{v})$. **1 point**.
- Re-write $(AB)(\mathbf{v}) = B(\lambda\mathbf{v})$ to get $A(B\mathbf{v}) = \lambda(B\mathbf{v})$. **1 point**.
- Prove that $B\mathbf{v} \neq 0$. **1 point**.
  By contradiction. If $B\mathbf{v}$ were 0 then $0 = (B^{-1}AB)(\mathbf{v}) = \lambda\mathbf{v}$, which would imply $\lambda = 0$ or $\mathbf{v} = 0$, which are both false statements. Hence, $B\mathbf{v} \neq 0$.
- Conclude that $B\mathbf{v}$ is an eigenvector of A associated to the eigenvalue λ. **1 point**.
  This follows from the definition of eigenvector and the previous item.